Supersymmetric Multi-trace Boundary Conditions in AdS

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Boundary conditions for massive fermions are investigated in AdS_d for $d \geq 2$. For fermion masses in the range $0 \leq |m| < 1/2\ell$ with ℓ the AdS length, the standard notion of normalizeability allows a choice of boundary conditions. As in the case of scalars at or slightly above the Breitenlohner-Freedman (BF) bound, such boundary conditions correspond to multi-trace deformations of any CFT dual. By constructing appropriate boundary superfields, for $d = 3, 4, 5$ we identify joint scalar/fermion boundary conditions which preserve either $\mathcal{N} = 1$ supersymmetry or $\mathcal{N} = 1$ superconformal symmetry on the boundary. In particular, we identify boundary conditions corresponding via AdS/CFT (at large N) to a 595-parameter family of double-trace marginal deformations of the low-energy theory of N M2-branes which preserve $\mathcal{N} = 1$ superconformal symmetry. We also establish that (at large N and large 't Hooft coupling λ) there are no marginal or relevant multi-trace deformations of 3+1 $\mathcal{N} = 4$ super Yang-Mills which preserve even $\mathcal{N} = 1$ supersymmetry.

Contents

I. INTRODUCTION

A central aspect of the Anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence [1, 2, 3] is that gauge-invariant deformations of the CFT Lagrangian correspond to modifications of the AdS boundary conditions. The requirement of gauge invariance can be implemented by constructing operators that transform in the adjoint representation of the gauge group and taking a trace. In this way, these deformations can be classified by the number of traces. Single-trace deformations, which in the CFT correspond to the addition of simple sources, correspond [3, 4] to fixing one of the Fefferman-Graham type coefficients [5] that control the fall-off of bulk fields at infinity. In contrast, multi-trace deformations correspond to imposing relations between two or more such coefficients [6].

Of course, one should choose boundary conditions that lead to a well-defined bulk theory. In particular, they should be compatible with the bulk inner product, ensuring that it is both finite and conserved. As observed in [4, 7], for tachyonic scalars near the Breitenlohner-Freedman bound, the standard scalar inner product is in fact compatible

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d	Fermion Field Content	# of Real Scalars	# of Real Supercharges
3	2 Majorana	2	2
4	1 Majorana	2	4
5	1 Dirac	4	8

TABLE I: Scalar supermultiplet content. Note that in $d = 5$, a complex Dirac spinor can be equivalently represented by a pair of symplectic Majorana spinors.

with a variety of boundary conditions; see [7, 8, 9, 10] for comments on the vector and tensor cases. It is also possible to consider more general inner products [10] (and see comments in [9]), though at the risk of sacrificing positivity and introducing ghosts. We shall therefore restrict attention to the standard inner products below[1].

Our goal here is to understand supersymmetric multi-trace boundary conditions for bulk scalar supermultiplets, generalizing certain results of [11] for massless multiplets in AdS_4. Supersymmetric single-trace boundary conditions in AdS_4 were analyzed in [7, 12, 13] (see also [14] for the corresponding analysis in AdS_2). Some AdS/CFT applications of multi-trace boundary conditions were given in [6, 15, 16]. Supersymmetric boundary conditions at finite boundaries were studied in [17].

We begin our study by investigating multi-trace boundary conditions for Dirac fermions of general mass m propagating on a fixed AdS_d background with $d \geq 2$. See [7] for some discussion of fermion boundary conditions in $d = 4$ and e.g., [18, 19, 20] for treatments of single-trace boundary conditions for fermions in the context of AdS/CFT. Our fermions are free aside from possible non-linear boundary conditions. We find unique boundary conditions for $|m| \geq \frac{1}{2}$, but a wide class of boundary conditions for $|m| < \frac{1}{2}$. One should be able to include bulk interactions using the techniques of either [21] or [11, 22], and based on the results of those works for scalars, one would not expect this to change the allowed boundary conditions[2].

We then specialize to the cases $d = 3, 4, 5$ to study supersymmetry. We study allowed boundary conditions for systems of scalars and spin-1/2 fermions as specified in Table I. For $d = 4, 5$ these systems admit $\mathcal{N} = 1$ bulk supersymmetry, while our $d = 3$ system admits $\mathcal{N} = (1, 0)$ supersymmetry[3]. As in [11], constructing appropriate boundary superfields from the Fefferman-Graham coefficients of bulk fields will allow us to identify boundary conditions preserving either the full supersymmetry (i.e., superconformal symmetry on the boundary) or a certain subalgebra (naturally called boundary Poincaré supersymmetry) containing half of the original supercharges. With our field content, non-trivial boundary conditions preserving boundary Poincaré supersymmetry are generally allowed for fermion mass $|m| < \frac{1}{2}$, though choices preserving superconformal invariance (and which correspond to an integer number of traces in a dual field theory) arise only for special values of m. As in [11], we find that supersymmetric boundary conditions always relate two distinct bulk scalar fields. As a result, the multi-trace boundary conditions allowed for a single scalar (and used in so-called designer gravity theories [23]) admit no supersymmetric generalization.

The plan of this paper is as follows. After stating our conventions in section I A, we describe the allowed boundary conditions in section II. Here we relegate technical details to the appendices: appendix A reviews solutions of the AdS_d Dirac equation following [24], and appendix B analyzes the convergence of the inner product. We then classify boundary conditions preserving supersymmetry as stated above for AdS_4 (section III), AdS_5 (section IV), and AdS_3 (section V). We close with some discussion in section VI, including comments on supersymmetric deformations of M2-brane theories [25, 26] and 3+1 $\mathcal{N} = 4$ super Yang-Mills.

[1] Strictly speaking, in the context of non-linear theories one should speak of the symplectic structure instead of the inner product. Since the symplectic structure is simply an (indefinite) inner product on the space of linearized fields, we will take this to be implied by our use of the term "inner product" without further comment.

[2] However, in special cases interactions do give rise to new logarithms which may break a conformal invariance that appears to be preserved in the linearized approximation.

[3] In fact, for $d = 3$ our field content is that of an $\mathcal{N} = (2, 0)$ supermultiplet, though we will only find non-trivial boundary conditions which preserve $\mathcal{N} = (1, 0)$.

A. Conventions

It is convenient to discuss boundary conditions using the conformal compactification of AdS spacetime. One may describe this compactification by starting with the global AdS metric

$$ds^2 = -(1+r^2)dt^2 + \frac{dr^2}{1+r^2} + r^2 d\Omega_{d-2}^2 \,. \tag{1.1}$$

Here $d\Omega_{d-2}^2$ is the line element of the unit sphere S^{d-2} and we have set the AdS length ℓ to one. Introducing the coordinate Ω via $r = \Omega^{-1} - \Omega/4$, one defines an unphysical metric $\tilde{g}_{ab} = \Omega^2 g_{ab}$ which satisfies

$$\widetilde{ds^2} = d\Omega^2 - \left(1 + \frac{1}{4}\Omega^2\right)^2 dt^2 + \left(1 - \frac{1}{4}\Omega^2\right)^2 d\Omega_{d-2}^2 \,. \tag{1.2}$$

The unphysical spacetime is thus a manifold with boundary $\mathcal{I} \cong \mathbb{R} \times S^{d-2}$ at $\Omega = 0$.

In this spacetime, $\tilde{n}_a = \tilde{\nabla}_a \Omega$ coincides with the unit normal to the boundary, where $\tilde{\nabla}_a$ is the torsion-free covariant derivative compatible with $\tilde{g}_{ab}$. It is also useful to define the orthogonal projector $\tilde{h}_{ab} = \tilde{g}_{ab} - \tilde{n}_a \tilde{n}_b$, which at $\Omega = 0$ becomes the induced metric on the boundary

$$\tilde{h}_{ab} dx^a dx^b \,|_{\mathcal{I}} = -dt^2 + d\Omega_{d-2}^2 \,; \tag{1.3}$$

i.e., the Einstein static universe. Indices on all tensor fields with a tilde are raised and lowered with the unphysical metric $\tilde{g}_{ab}$ and its inverse $\tilde{g}^{ab}$.

Our conventions for treating spinors are as follows. Spacetime indices are denoted by $a, b, \ldots$, while indices on a flat internal space are denoted by $\hat{a}, \hat{b}, \ldots = \hat{0}, \hat{1}, \hat{2}, \ldots$. In d spacetime dimensions, the Dirac spinor representation is $2^{[d/2]}$ dimensional, where $[x]$ is the integer part of x. The flat-space gamma matrices are $2^{[d/2]} \times 2^{[d/2]}$ matrices satisfying

$$\{\gamma_{\hat{a}}, \gamma_{\hat{b}}\} = 2\eta_{\hat{a}\hat{b}} \,, \tag{1.4}$$

where $\eta_{\hat{a}\hat{b}}$ is the metric of Minkowski space with signature $(-++\ldots)$. We also note that $(\gamma^{\hat{0}})^\dagger = -\gamma^{\hat{0}}$ and $(\gamma^{\hat{k}})^\dagger = \gamma^{\hat{k}}$, with $\hat{k} = \hat{1}, \hat{2}, \ldots$. For a given spacetime metric g_{ab}, we can define an orthonormal frame $\{e^{\hat{a}}{}_a\}$ which satisfies $e^{\hat{a}}{}_a e^{\hat{b}}{}_b \eta_{\hat{a}\hat{b}} = g_{ab}$. The curved space gamma matrices are then given by $\gamma_a = e^{\hat{a}}{}_a \gamma_{\hat{a}}$ and satisfy $\gamma_{(a}\gamma_{b)} = g_{ab}$. Below, we take our covariant derivative ∇_a to act on spinors as

$$\nabla_a \psi = \partial_a \psi + \Gamma_a \psi \quad \text{where} \quad \Gamma_a = \frac{1}{4}\,\omega_a{}^{\hat{a}\hat{c}} \gamma_{[\hat{a}}\gamma_{\hat{c}]} \quad \text{and} \quad -de^{\hat{a}} = \omega^{\hat{a}}{}_{\hat{c}} \wedge e^{\hat{c}} \,. \tag{1.5}$$

Here Γ_a is the spin connection and $\omega_a{}^{\hat{a}\hat{c}}$ are the rotation coefficients. We assume that all spinors are anticommuting, and define the Dirac conjugate of a spinor ψ to be $\overline{\psi} = \psi^\dagger \gamma^{\hat{0}}$. Tildes will denote quantities defined analogously in terms of $\tilde{g}_{ab}$; e.g., $\tilde{\gamma}_{(a}\tilde{\gamma}_{b)} = \tilde{g}_{ab}$.

II. BOUNDARY CONDITIONS FOR SCALARS AND FERMIONS

Let us consider free theories of scalars and Dirac fermions propagating on a fixed AdS$_d$ background. For a set of scalars ϕ_I of masses m_{ϕ_I} and Dirac fermions $\psi_{\hat{I}}$ of masses $m_{\psi_{\hat{I}}}$, the Lagrangian is

$$L = \sum_I \left(-\frac{1}{2}\nabla^a \phi_I \nabla_a \phi_I - \frac{1}{2} m_{\phi_I}^2 \phi_I^2\right) + \sum_{\hat{I}} i\left[\frac{1}{2}\left(\overline{\psi}_{\hat{I}} \gamma^a \nabla_a \psi_{\hat{I}} - \overline{\nabla_a \psi}_{\hat{I}} \gamma^a \psi_{\hat{I}}\right) - m_{\psi_{\hat{I}}} \overline{\psi}_{\hat{I}} \psi_{\hat{I}}\right] . \tag{2.1}$$

We wish to identify boundary conditions for which the standard inner product is both finite and conserved on the space of linearized solutions. These conditions suffice to yield a well-defined phase space. In particular, they imply that charges corresponding to the AdS isometries are well-defined and conserved. They also ensure that the linearized quantum theory evolves unitarily.

Since we consider only linear fields and use the standard inner product, we may identify normalizeable modes separately for each field. We first briefly recall the results for scalar fields using the Klein-Gordon inner product. Denoting the Breitenlohner-Freedman bound by $m_{BF}^2 = -\frac{(d-1)^2}{4}$, one finds two cases. For $m_{\phi_I}^2 \geq m_{BF}^2 + 1$, there is a

unique complete set of normalizeable modes, and any other (non-normalizeable) modes must be fixed by the boundary conditions.

In contrast, much more general boundary conditions are allowed for $m^2_{\phi_I} < m^2_{BF} + 1$, though the case $m^2_{\phi_I} < m^2_{BF}$ is usually ignored due to instabilities. The boundary conditions are most simply expressed in terms of a Fefferman-Graham-type expansion [5] of ϕ_I. For $m^2_{\phi_I} \neq m^2_{BF}$ we have[4]

$$\phi_I = \alpha_I \Omega^{\lambda_{I,-}} + \beta_I \Omega^{\lambda_{I,+}} + \ldots, \quad \text{where} \quad \lambda_{I,\pm} = \frac{d-1 \pm \sqrt{(d-1)^2 + 4m^2_{\phi_I}}}{2}\,. \tag{2.2}$$

Here α_I, β_I are independent of Ω, but can depend on time and angles on the sphere. For $m^2_{\phi_I} = m^2_{BF}$, the roots (2.2) are degenerate and the solution becomes

$$\phi_I = \alpha_I \Omega^{\lambda} \log \Omega + \beta_I \Omega^{\lambda} + \ldots \quad \text{where} \quad \lambda = (d-1)/2. \tag{2.3}$$

We refer to α_I, β_I as the boundary fields corresponding to the bulk field ϕ_I.

Normalizeability places no restriction on the boundary fields, but we must still impose conservation. Considering two linearized solutions $\delta_{1,2}\phi_I$ and the corresponding $\delta_{1,2}\alpha_I, \delta_{1,2}\beta_I$, a short calculation shows that the flux through the boundary is

$$\mathcal{F}^{\phi} = \sum_I \int_{\mathcal{I}} (\lambda_{I,+} - \lambda_{I,-})\,(\delta_1\beta_I\delta_2\alpha_I - \delta_2\beta_I\delta_1\alpha_I)\, d^{d-1}S\,, \tag{2.4}$$

where $d^{d-1}S$ is the integration element on $\mathcal{I}$. This flux vanishes precisely when the boundary conditions restrict α_I, β_I to a "Lagrange submanifold" in the space of possible (α_I, β_I). Such boundary conditions can often be (locally) specified by choosing a function[5] $W(\alpha_I, x)$ and requiring

$$(\lambda_{I,+} - \lambda_{I,-})\beta_I(x) = \frac{\partial W}{\partial \alpha_I}\,, \tag{2.5}$$

where here $x \in \mathcal{I}$. For theories with a dual CFT, this boundary condition corresponds to adding a multi-trace term $W(\mathcal{O}_I, x)$ to the field theory Lagrangian, where $\mathcal{O}_I$ is the operator dual to ϕ_I for $W = 0$ boundary conditions. Certain exceptional cases that will arise in later sections can be constructed as limits of (2.5), but for our purposes may be better described by choosing a W that depends on both α's and β's. As an example, with $I = 1, 2$ we might choose $W = W(\alpha_1, \beta_2)$ and take

$$(\lambda_{2,+} - \lambda_{2,-})\alpha_2 = -\frac{\partial W}{\partial \beta_2}\,, \quad (\lambda_{1,+} - \lambda_{1,-})\beta_1 = \frac{\partial W}{\partial \alpha_1}\,. \tag{2.6}$$

Such boundary conditions again correspond to adding W to the dual CFT Lagrangian of the $W = 0$ theory.

The corresponding analysis for Dirac fermions (with the standard inner product) is performed in appendix B. We make heavy use of [24], which solves the massive Dirac equation in AdS$_d$ (see appendix A for a review). For $|m_{\psi_{\hat{I}}}| \geq 1/2$, there is a unique complete set of normalizeable modes. Any other non-normalizeable modes must be fixed by the boundary condition. In contrast, much more general boundary conditions are allowed for $|m_{\psi_{\hat{I}}}| < 1/2$.

For this latter case it is again convenient to introduce boundary fields. In appendix B, we derive the asymptotic expansion

$$\psi_{\hat{I}} = \alpha^{\psi}_{\hat{I}}\, \Omega^{\frac{d-1}{2} - m_{\psi_{\hat{I}}}} + \beta^{\psi}_{\hat{I}}\, \Omega^{\frac{d-1}{2} + m_{\psi_{\hat{I}}}} + {\alpha'}^{\psi}_{\hat{I}}\, \Omega^{\frac{d+1}{2} - m_{\psi_{\hat{I}}}} + {\beta'}^{\psi}_{\hat{I}}\, \Omega^{\frac{d+1}{2} + m_{\psi_{\hat{I}}}} + O(\Omega^{\frac{d+3}{2} - |m_{\psi_{\hat{I}}}|})\,. \tag{2.7}$$

Here $\alpha^{\psi}_{\hat{I}}, \beta^{\psi}_{\hat{I}}$ are again boundary fields depending only on time and angles on the S^{d-2}. For later convenience we have included certain sub-leading terms whose coefficients ${\alpha'}^{\psi}_{\hat{I}}, {\beta'}^{\psi}_{\hat{I}}$ are determined by $\alpha^{\psi}_{\hat{I}}, \beta^{\psi}_{\hat{I}}$. The coefficients satisfy

$$\tilde{P}_+ \alpha^{\psi}_{\hat{I}} = 0\,, \quad \tilde{P}_- \alpha^{\psi}_{\hat{I}} = \alpha^{\psi}_{\hat{I}}, \quad {\alpha'}^{\psi}_{\hat{I}} \;=\; -\frac{1}{1 - 2m_{\psi_{\hat{I}}}}\, \tilde{h}^{ab} \tilde{\gamma}_a \tilde{\nabla}_b \alpha^{\psi}_{\hat{I}}\,, \tag{2.8}$$

[4] Including gravitational backreaction would modify this expansion [21, 22] if $4\lambda_- \leq (d-1)$, which can only be satisfied for $d \leq 4$. To include backreaction while avoiding this regime, one would have to restrict the range of m_{ϕ_I}, though this is most likely only a technical complication [21, 22]. In any case, we ignore such backreaction here and consider propagation on a fixed spacetime.

[5] We assume throughout this work that W does not involve derivatives of fields along the boundary (see [10, 27] for subtleties that arise when W involves time derivatives).

$$\tilde{P}_{-}\beta^{\psi}_{\hat{I}} = 0\,, \quad \tilde{P}_{+}\beta^{\psi}_{\hat{I}} = \beta^{\psi}_{\hat{I}}\,, \quad \beta'^{\psi}_{\hat{I}} \;=\; \frac{1}{1+2m_{\psi_{\hat{I}}}}\,\tilde{h}^{ab}\tilde{\gamma}_a\tilde{\nabla}_b\beta^{\psi}_{\hat{I}}\,, \tag{2.9}$$

where we have defined the radial projectors $\tilde{P}_{\pm} = \frac{1}{2}(1 \pm \tilde{n}_a\tilde{\gamma}^a)$. Note that when $m_{\psi_{\hat{I}}} < 0$, the $\beta^{\psi}_{\hat{I}}$ term in (2.7) is actually the leading term in the asymptotic expansion.

It remains to impose conservation. Inserting the asymptotic expansion (2.7) into the fermion inner product (see appendix B) and using (2.8), (2.9), we find the fermionic contribution to the flux through the boundary

$$\mathcal{F}^{\psi} = i\sum_{\hat{I}}\int_{\mathcal{I}}\left[\left(\overline{\delta_1\alpha^{\psi}_{\hat{I}}}\,\delta_2\beta^{\psi}_{\hat{I}} - \overline{\delta_1\beta^{\psi}_{\hat{I}}}\,\delta_2\alpha^{\psi}_{\hat{I}}\right) - \left(\delta_1 \leftrightarrow \delta_2\right)\right] d^{\,d-1}S\,. \tag{2.10}$$

Recall, however, that we are interested in theories of the form (2.1) which contain both scalars and fermions. In the pure scalar case, it was not necessary for the flux to vanish separately for each scalar field. Instead, only the total flux was required to vanish. Similarly, when both scalars and fermions are present, it is only the total flux $\mathcal{F} \equiv \mathcal{F}^{\phi} + \mathcal{F}^{\psi}$ involving both types of fields that must vanish. This occurs when the boundary conditions restrict the fields to what one may call a "Lagrange submanifold" in the space of all $(\alpha_I, \beta_I, \alpha^{\psi}_{\hat{I}}, \beta^{\psi}_{\hat{I}})$. Again, this is often locally equivalent to choosing a real function $W(\alpha_I, \alpha^{\psi}_{\hat{I}}, \overline{\alpha^{\psi}_{\hat{I}}}, x)$ and defining

$$(\lambda_{I,+} - \lambda_{I,-})\beta_I(x) = \frac{\partial W}{\partial \alpha_I}\,, \quad -i\beta^{\psi}_{\hat{I}}(x) = \frac{\partial W}{\partial \overline{\alpha^{\psi}_{\hat{I}}}}\,. \tag{2.11}$$

In other situations, one may wish to choose W to depend on α's and β's in analogy with the scalar case. In each instance, this corresponds to deforming the $W = 0$ dual CFT by adding W to its action. For simplicity, we now assume that all scalars satisfy $m^2_{\phi_I} < m^2_{BF} + 1$ and all fermions satisfy $|m_{\psi_{\hat{I}}}| < 1/2$.

As a brief example, consider a theory with one Dirac fermion and no scalars and suppose that we desire a linear boundary condition which respects local Lorentz-invariance and translation-invariance on the boundary. In even dimensions we must impose

$$\beta^{\psi} = iq\gamma_{d+1}\alpha^{\psi} \quad \text{where} \quad \gamma_{d+1} = \frac{i^{\frac{d-2}{2}}}{d!}\epsilon^{a_1\ldots a_d}\gamma_{a_1}\ldots\gamma_{a_d} = i^{\frac{d-2}{2}}\gamma^{\hat{0}}\gamma^{\hat{1}}\ldots\gamma^{\widehat{d-1}}\,, \tag{2.12}$$

for some real q. Here γ_{d+1} is the analogue of γ_5 in $d = 4$ and satisfies

$$\{\gamma_{d+1}, \gamma^a\} = 0\,, \quad \gamma^{\dagger}_{d+1} = \gamma_{d+1}\,, \quad (\gamma_{d+1})^2 = 1\,. \tag{2.13}$$

The Breitenlohner-Freedman boundary conditions for $d = 4$ [7] correspond to the particular choices $q = 0$ or $q = \infty$. Since $\gamma_{d+1}\tilde{P}_{\pm} = \tilde{P}_{\mp}\gamma_{d+1}$, the boundary condition (2.12) is consistent with (2.8), (2.9). In contrast, in odd dimensions, the matrix γ_{d+1} is proportional to the identity and commutes with $\tilde{P}_{\pm}$ so that (2.12) implies $\alpha^{\psi} = \beta^{\psi} = 0$. Thus, in odd dimensions this theory has no non-trivial linear boundary conditions with the desired properties. However, as shown below, there are more possibilities with a greater number of fermions.

As a final comment we mention that, at least for linearized fields, the above analysis is equivalent to studying self-adjoint extensions of the spatial wave operator. Such an approach was applied to massive scalar fields and massless vector and tensor fields in [8]. The authors showed that a simple 2-parameter family of wave operators sufficed to describe all of these fields (though there is some subtlety associated with the choice of inner product in the tensor case, see [10]). As is briefly mentioned in appendix B, this approach can also be used for our fermions, and the analysis again reduces to the wave operator studied in [8]. Comparison with [8] explicitly shows that stability issues of the sort that would arise for scalars with $m^2_{\phi} < m^2_{BF}$ cannot occur for Dirac fermions with real mass m_{ψ}. Indeed, the relevant inequality is $(m_{\psi} \pm \frac{1}{2})^2 \geq 0$. Thus, in some sense $m_{\psi} = \pm\frac{1}{2}$ is analogous to saturating the Breitenlohner-Freedman bound, even though the boundary conditions are unique for $|m_{\psi}| \geq \frac{1}{2}$.

III. $\mathcal{N} = 1$ SUPERSYMMETRY IN $d = 4$

Consider the AdS$_4$ theory of a Majorana fermion $\hat{\psi}$ and two real scalars (A, B) with Lagrangian (2.1). To match the usual normalization of the action for Majorana fermions we define $\psi \equiv \sqrt{2}\,\hat{\psi}$ and work exclusively with this rescaled spinor. The fermion ψ obeys the Majorana condition $\overline{\psi} = \psi^T C$, where C is the charge conjugation matrix and satisfies

$$C\gamma^a C^{-1} = -(\gamma^a)^T\,, \quad C^T = C^{-1} = C^{\dagger} = -C\,. \tag{3.1}$$

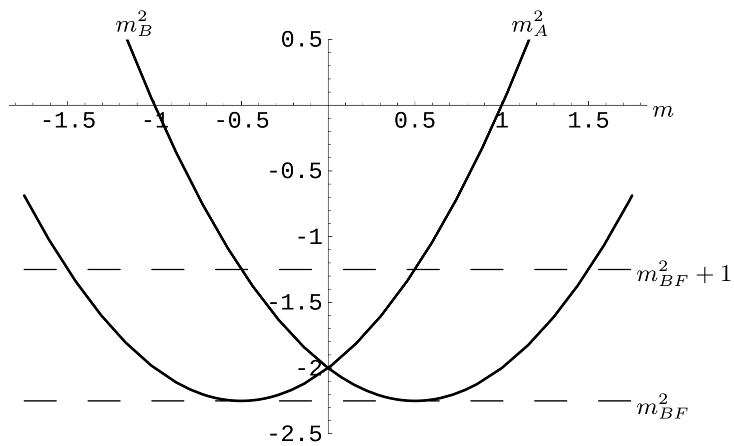

FIG. 1: The scalar masses m_A^2, m_B^2 (solid curves) are plotted against the fermion mass m. Dashed lines mark the window between m_{BF}^2 and m_{BF}^2+1.

When the scalars masses (m_A, m_B) and fermion mass (m) are related by

$$m_A^2 = m^2 + m - 2, \quad m_B^2 = m^2 - m - 2\,, \tag{3.2}$$

the action is invariant [7, 28] under the $\mathcal{N}=1$ supersymmetry transformations

$$\delta_\eta A = \tfrac{i}{\sqrt{2}}\,\overline{\eta}\psi, \qquad \delta_\eta B = -\tfrac{1}{\sqrt{2}}\,\overline{\eta}\gamma_5\psi \tag{3.3}$$

$$\delta_\eta\psi = -\tfrac{1}{\sqrt{2}}\,\left[\gamma^a\nabla_a(A+i\gamma_5 B) + (m-1)A + i(m+1)\gamma_5 B\right]\eta \quad , \tag{3.4}$$

where the supersymmetry-generating parameter η is a Killing spinor, i.e.

$$\left(\nabla_a + \frac{1}{2}\gamma_a\right)\eta = 0\,. \tag{3.5}$$

The case $m=0$ was studied in [11]; we closely follow their analysis and correct certain equations below.

An important feature of supersymmetry in anti-de Sitter space (see e.g. [29]) is that fields in the same multiplet do not necessarily have the same mass, though the degeneracy is restored in the flat space limit $\ell^{-1}\to 0$. The scalar masses are plotted in Fig. 1. Notable features of the relation (3.2) are as follows: i) massless fermions correspond to conformally coupled scalar fields, $m_A^2 = m_B^2 = -2$; ii) $m_{A,B}^2$ has a global minimum of $-9/4 = m_{BF}^2$ at $m=\mp 1/2$, so the scalars always satisfy the Breitenlohner-Freedman bound; iii) $m_{A,B}^2$ reaches the value m_{BF}^2+1 at $m=\pm 1/2$. Thus the range $|m|<1/2$, which we have seen allows general boundary conditions for fermions, typically matches the mass range that allows general boundary conditions for scalars. The one exception occurs for $m=\pm 1/2$ where one scalar saturates the BF bound and the other has squared mass m_{BF}^2+1. The analogous properties also hold for the AdS$_5$, AdS$_3$ cases studied in sections IV and V. We restrict attention to the case $|m|<1/2$ below.

Solutions to the Killing spinor equation [7, 30] have leading terms as $\Omega\to 0$ given by

$$\eta = \eta_+\Omega^{-1/2} + \frac{1}{2}\eta_-\Omega^{1/2} + \ldots\,, \tag{3.6}$$

where $\eta_\pm = \tilde{P}_\pm U$ and U is a constant spinor. Using the asymptotic expansions of A, B, ψ, η from (2.2),(2.7),(3.6) in the supersymmetry transformations and matching terms order by order in Ω gives the action of supersymmetry on our boundary fields:

$$\delta_\eta\alpha_A = \frac{i}{\sqrt{2}}\,\overline{\eta_+}\,\alpha_\psi \tag{3.7}$$

$$\delta_\eta\beta_A = \frac{i}{2\sqrt{2}}\,\overline{\eta_-}\,\beta_\psi + \frac{1}{1+2m}\frac{i}{\sqrt{2}}\,\overline{\eta_+}\,\tilde{h}^{ab}\tilde{\gamma}_a\tilde{\nabla}_b\beta_\psi \tag{3.8}$$

$$\delta_\eta\alpha_B = -\frac{1}{\sqrt{2}}\,\overline{\eta_+}\,\gamma_5\beta_\psi \tag{3.9}$$

$$\delta_\eta\beta_B = -\frac{1}{2\sqrt{2}}\,\overline{\eta_-}\,\gamma_5\alpha_\psi + \frac{1}{1-2m}\frac{1}{\sqrt{2}}\,\overline{\eta_+}\,\gamma_5\tilde{h}^{ab}\tilde{\gamma}_a\tilde{\nabla}_b\alpha_\psi \tag{3.10}$$

$$\delta_\eta \alpha_\psi = -\frac{1}{\sqrt{2}}\left[(m-1)\alpha_A\eta_- - i(1-2m)\gamma_5\beta_B\eta_+ + \tilde{h}^{ab}\tilde{\gamma}_a\tilde{\nabla}_b\alpha_A\eta_+\right] \tag{3.11}$$

$$\delta_\eta \beta_\psi = -\frac{1}{\sqrt{2}}\left[i(m+1)\gamma_5\alpha_B\eta_- + (1+2m)\beta_A\eta_+ - i\gamma_5\tilde{h}^{ab}\tilde{\gamma}_a\tilde{\nabla}_b\alpha_B\eta_+\right]. \tag{3.12}$$

We now consider a 2-dimensional space of the 4 linearly independent Killing spinors η associated with some choice of Poincaré coordinates for AdS:

$$ds^2 = \frac{1}{z^2}\left(-dt^2 + dz^2 + dx_1^2 + dx_2^2\right), \quad z \geq 0. \tag{3.13}$$

In general, the conformally rescaled Killing spinor equation is

$$\left(\tilde{\nabla}_a + \Omega^{-1}\tilde{\gamma}_a\tilde{P}_-\right)\tilde{\eta} = 0, \tag{3.14}$$

where $\tilde{\eta} = \Omega^{1/2}\eta$. For the metric (3.13), we choose the conformal factor $\Omega = z$. Solutions to (3.14) are then $\eta = \Omega^{-1/2}\varepsilon_+$, where ε_+ is a constant spinor satisfying $\tilde{P}_-\varepsilon_+ = 0$ and $\tilde{P}_\pm = \frac{1}{2}(1 \pm \gamma_z)$, with γ_z a flat space gamma matrix. For this two-dimensional space of Killing spinors, the transformations (3.7)-(3.12) simplify somewhat, and anti-commutators of such transformations generate the manifest Poincaré symmetries of (3.13).

To further simplify the supersymmetry transformations, note that the 4-component bulk Majorana spinors $\alpha_\psi, \beta_\psi, \varepsilon_+$ satisfy projection conditions defined by $\tilde{P}_\pm$. As a result, they define real, 2-component spinors living on the boundary $\mathbb{R}^3$. To make this explicit, let the indices i, j run over t, x_1, x_2. Then the matrices $\underline{\Gamma}^j = i\tilde{\gamma}^j\gamma_5$ satisfy $\{\underline{\Gamma}^i, \underline{\Gamma}^j\} = \eta^{ij}$, $[\underline{\Gamma}^j, \tilde{P}_\pm] = 0$, and so form a representation of the 3-dimensional Clifford algebra on the boundary. This is conveniently realized in terms of the real 2×2 matrices $\underline{\gamma}^t = i\sigma^2, \underline{\gamma}^{x_1} = \sigma^1, \underline{\gamma}^{x_2} = \sigma^3$, where $\vec{\sigma}$ denotes the Pauli matrices (A10). Our two-component spinor conventions are as follows. Spinor indices are denoted by Greek letters $\kappa, \lambda, \ldots = 1, 2$. Spinor indices are raised and lowered with the antisymmetric tensors $\epsilon^{\kappa\lambda}, \epsilon_{\kappa\lambda}$, which we define by $\epsilon^{12} = 1 = -\epsilon^{21}, \epsilon_{\kappa\lambda} = -\epsilon^{\kappa\lambda}$. Then

$$\psi^\kappa = \epsilon^{\kappa\lambda}\psi_\lambda, \quad \psi_\kappa = \epsilon_{\kappa\lambda}\psi^\lambda \tag{3.15}$$

and the spinor product is

$$\chi\psi \equiv \chi^\lambda\psi_\lambda = -\chi_\lambda\psi^\lambda = \psi^\lambda\chi_\lambda = \psi\chi. \tag{3.16}$$

In this standard notation, repeated spinor indices are summed over $1, 2$. Note that $(\underline{\gamma}^t)^{\kappa\lambda} = \epsilon^{\kappa\lambda}$, so for real spinors $\overline{\psi}\chi = \psi^T\underline{\gamma}^t\chi = -\psi\chi$. Also, the three-dimensional Majorana condition $\bar{\psi} = \psi^T C$ reduces exactly to the reality condition $\psi = \psi^*$, where the three-dimensional charge conjugation matrix is $C = \underline{\gamma}^t$. Lastly, for Majorana spinors one has

$$\overline{\psi}\chi = \overline{\chi}\psi, \quad \overline{\psi}\underline{\gamma}^j\chi = -\overline{\chi}\underline{\gamma}^j\psi. \tag{3.17}$$

Using these results, the Poincaré supersymmetries defined by $\eta = \Omega^{1/2}\varepsilon_+$ may be written

$$\delta_\varepsilon\alpha_A = \frac{1}{\sqrt{2}}\varepsilon_+ a_\psi \tag{3.18}$$

$$(1+2m)\delta_\varepsilon\beta_A = -\frac{1}{\sqrt{2}}\varepsilon_+\underline{\gamma}^j\partial_j\beta_\psi \tag{3.19}$$

$$\delta_\varepsilon\alpha_B = -\frac{1}{\sqrt{2}}\varepsilon_+\beta_\psi \tag{3.20}$$

$$(1-2m)\delta_\varepsilon\beta_B = -\frac{1}{\sqrt{2}}\varepsilon_+\underline{\gamma}^j\partial_j a_\psi \tag{3.21}$$

$$\delta_\varepsilon a_\psi = -\frac{1}{\sqrt{2}}\left[(1-2m)\beta_B\varepsilon_+ - \underline{\gamma}^j\partial_j\alpha_A\varepsilon_+\right] \tag{3.22}$$

$$\delta_\varepsilon\beta_\psi = -\frac{1}{\sqrt{2}}\left[(1+2m)\beta_A\varepsilon_+ + \underline{\gamma}^j\partial_j\alpha_B\varepsilon_+\right], \tag{3.23}$$

where a_ψ denotes the two-component boundary spinor defined by $i\gamma_5\alpha_\psi$.

Under (3.18)-(3.23), boundary fields mix only within each of the disjoint sets $(\alpha_A, \alpha_\psi, \beta_B)$, $(\alpha_B, \beta_\psi, \beta_A)$. We may therefore construct useful boundary superfields from each set separately. To do so, introduce a real anti-commuting 2-component spinor θ^λ and define

$$\Phi_- = \alpha_A + \overline{\theta} a_\psi + \frac{1}{2}\overline{\theta}\theta(1-2m)\beta_B, \quad \text{and} \quad \Phi_+ = \alpha_B - \overline{\theta}\beta_\psi - \frac{1}{2}\overline{\theta}\theta(1+2m)\beta_A \,. \tag{3.24}$$

Taking θ to have conformal dimension $-1/2$, we note that the superfield $\Phi_\pm$ has a well-defined conformal dimension $1 \pm m$. One may now check that (3.18)-(3.23) can be written as

$$\delta_\varepsilon \Phi_\pm = \frac{1}{\sqrt{2}} \left[-\varepsilon_+^\kappa \frac{\partial}{\partial \theta^\kappa} + \varepsilon_+ \underline{\gamma}^j \theta \partial_j \right] \Phi_\pm \tag{3.25}$$

and that δ_ε acts in precisely the same way on $\delta_\varepsilon \Phi_\pm$; i.e., the $\Phi_\pm$ are indeed superfields and (3.25) defines a covariant derivative on superspace. Finally, using the above relations and the two-component spinor identity $(\theta\psi)(\theta\chi) = -\frac{1}{2}(\theta\theta)\psi\chi$, the total flux $\mathcal{F}^\phi + \mathcal{F}^\psi$ can be written

$$\mathcal{F} = \int_{\mathcal{I}} d^3S \int d^2\theta \left(\delta_1 \Phi_- \delta_2 \Phi_+ - \delta_1 \Phi_+ \delta_2 \Phi_- \right) \,. \tag{3.26}$$

It is now clear that for any function W, the boundary condition

$$\Phi_- = \frac{\delta W(\Phi_+)}{\delta \Phi_+} \tag{3.27}$$

conserves the inner product and is invariant under the Poincaré supersymmetries. Such boundary conditions correspond to deformations of a dual CFT action by the addition of a term $\int_{\mathcal{I}} d^3S \int d^2\theta W$.

In terms of component fields we have

$$\alpha_A = W'(\alpha_B) \tag{3.28}$$

$$(1-2m)\beta_B = -(1+2m)W''(\alpha_B)\beta_A + \frac{1}{2}W'''(\alpha_B)\overline{\beta_\psi}\gamma_5\beta_\psi \tag{3.29}$$

$$\alpha_\psi = iW''(\alpha_B)\gamma_5\beta_\psi \,. \tag{3.30}$$

For general W, these boundary conditions break the conformal (and thus superconformal) symmetry; however, for the special choice

$$W(\Phi_+) = \frac{1+m}{2} \, q \, (\Phi_+)^{\frac{2}{1+m}} \tag{3.31}$$

the full $O(3,2)$ symmetry is preserved and, as one can explicitly check, so is the full supersymmetry defined by (3.7-3.12) for the arbitrary Killing spinors η. I.e., superconformal symmetry is preserved on the boundary.

As a simple example, consider the linear boundary condition $\Phi_- = q\Phi_+$. In terms of the component fields this is

$$\alpha_A = q\alpha_B \,, \quad (1-2m)\beta_B = -q(1+2m)\beta_A \,, \quad \alpha_\psi = iq\gamma_5\beta_\psi \,. \tag{3.32}$$

The boundary conditions of [7] correspond to $q = 0$ or $q = \infty$. Note that (3.32) and, more generally, (3.28) relate the two scalar fields to each other (α_A to α_B), (β_A to β_B). Boundary conditions of the sort studied in e.g. [6, 23] which relate α_A to β_A cannot be supersymmetrized to respect Poincaré supersymmetry on the boundary.

IV. $\mathcal{N} = 1$ SUPERSYMMETRY IN $d = 5$

For $d = 5$ we will consider a theory with four real scalars and a single Dirac spinor written as a symplectic Majorana pair. To set up the notation for symplectic spinors, we choose the five-dimensional gamma matrix representation $\gamma^{\hat{0}} = \begin{pmatrix} 0 & -iI_2 \\ -iI_2 & 0 \end{pmatrix}$, $\gamma^{\hat{k}} = \begin{pmatrix} 0 & -i\sigma^k \\ i\sigma^k & 0 \end{pmatrix}$, $\gamma^{\hat{4}} = \begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix}$ where $k = 1, 2, 3$. We also define the charge conjugation matrix $C = \begin{pmatrix} i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix}$, which satisfies $C^{-1} = C^\dagger = C^T = -C$ and $C\gamma^a C^{-1} = +(\gamma^a)^T$. The sign difference

between this last relation and (3.1) means that one cannot consistently define Majorana spinors in $d = 5$. Instead, one can impose a modified or "symplectic" Majorana condition (see e.g. [31])

$$\overline{\psi}^i = \psi_j^T \lambda^{ji} C \,, \quad i,j = 1,2 \tag{4.1}$$

where the Dirac conjugate is

$$\overline{\psi}^i \equiv \psi_i^\dagger \gamma^{\hat{0}} \tag{4.2}$$

and λ is the symplectic matrix

$$\lambda = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \quad \lambda^2 = -1, \quad \lambda^T = -\lambda. \tag{4.3}$$

The symplectic indices are raised and lowered by λ, with the convention

$$\psi^i = \psi_j \lambda^{ji} = -\psi_j \lambda^{ij} \,, \quad \lambda^{ij} = \lambda_{ij} \,. \tag{4.4}$$

With these definitions one can obtain the useful symplectic Majorana flip formulas

$$\overline{\psi}^i \chi_j = -\lambda_{jk}\lambda_{il}\overline{\chi}^k \psi_l \,, \quad \overline{\psi}^i \gamma^a \chi_j = -\lambda_{jk}\lambda_{il}\overline{\chi}^k \gamma^a \psi_l \,. \tag{4.5}$$

Now, consider a Dirac spinor ψ in $d = 5$. The fields $\psi_1 \equiv \psi \,, \psi_2 \equiv -\gamma^{\hat{0}} C \psi_1^*$ form a symplectic Majorana pair satisfying (4.1). The equations of motion are

$$\gamma^a \nabla_a \psi_1 - m\psi_1 = 0 \,, \quad \gamma^a \nabla_a \psi_2 + m\psi_2 = 0 \,, \quad \text{or} \quad \gamma^a \nabla_a \psi_i - m M_{ij} \psi_j = 0 \,, \tag{4.6}$$

where $M \in \mathrm{USp}(2)$ is given by

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{4.7}$$

In [32], the author considers theories with M a general element of USp(2), but for our purposes it will be sufficient to consider the simplest case when the fermion mass matrix is diagonal. This last equation of motion can also be obtained directly from the Lagrangian

$$L = -\frac{i}{2}\left(\overline{\psi}^i \gamma^a \nabla_a \psi_i - m M_{ij} \overline{\psi}^i \psi_j \right) , \tag{4.8}$$

where the above reality condition relates ψ_1 and ψ_2.

In five dimensions, a (complex) Dirac spinor has four degrees of freedom on-shell, and so via supersymmetry should correspond to four real scalars. Let us then consider a theory [32] of our standard form (2.1) where $I = 1, \ldots 4$. We take just one Dirac fermion, which we think of as being described by the symplectic Lagrangian (4.8) (note the extra overall minus sign relative to (2.1) inserted to match certain conventions of [32]). If the masses are related by

$$m_1^2 = m_2^2 = m^2 + m - \frac{15}{4}, \qquad m_3^2 = m_4^2 = m^2 - m - \frac{15}{4} \,, \tag{4.9}$$

the action is invariant under the $\mathcal{N} = 1$ supersymmetry transformations

$$\delta_\eta \phi^I \;=\; -i(\sigma^I \lambda)_{ij} \overline{\eta}^i \psi_j \tag{4.10}$$

$$\delta_\eta \psi_i \;=\; (\lambda \sigma^I)_{ji} \gamma^a \nabla_a \phi^I \eta_j + \left(3/2\,(\sigma^I)^T \lambda M + m M \lambda (\sigma^I)^\dagger\right)_{ij} \eta_j \phi^I \,. \tag{4.11}$$

Here $\sigma^I = (\vec{\sigma}, iI_2)$ and η_i is a symplectic Majorana Killing spinor [32] satisfying

$$\nabla_a \eta_i + \frac{1}{2}\gamma_a M_{ij} \eta_j = 0 \,. \tag{4.12}$$

As in the four-dimensional case, massless fermions correspond to conformally coupled scalar fields, $m_I^2 = -15/4$, and the scalars always satisfy the BF bound, $m_I^2 \geq m_{BF}^2 = -4$. For $I = 1,2$ ($I = 3,4$), the BF bound is saturated at

$m = -1/2$ ($m = 1/2$), and the scalar mass $m_{BF}^2 + 1$ is reached at $m = 1/2$ ($m = -1/2$). As usual, we consider only the range $-1/2 < m < 1/2$ below.

We again take the fermions ψ_i to have the asymptotic form (2.7). For $i = 1$, the expansion coefficients satisfy the relations (2.8), (2.9). For $i = 2$ however, these relations are slightly modified due to the opposite sign of the mass term in the Dirac equation (4.6). In particular, we note that $\tilde{P}_- \alpha_2^\psi = 0, \tilde{P}_+ \beta_2^\psi = 0$ and that certain sign changes occur in the expressions for $\alpha_2'^\psi, \beta_2'^\psi$. It is these properties under the radial projectors that allow us to have non-trivial boundary conditions for fermions in this odd dimensional theory, even though γ_{d+1} is proportional to the identity. We see that we can consistently relate α_1^ψ to β_2^ψ and α_2^ψ to β_1^ψ, since they are the same "type" of spinor. Solutions to the Majorana Killing spinor equation have leading terms as $\Omega \to 0$ given by

$$\eta_i = \alpha_i^\eta \Omega^{-1/2} + \frac{1}{2} \beta_i^\eta \Omega^{1/2} + \dots \tag{4.13}$$

where

$$\tilde{P}_- \alpha_1^\eta = 0 \,, \quad \tilde{P}_+ \beta_1^\eta = 0, \quad \tilde{P}_+ \alpha_2^\eta = 0 \,, \quad \tilde{P}_- \beta_2^\eta = 0 \,. \tag{4.14}$$

Inserting the asymptotic expansions of ϕ^I, ψ_i, η_i into the supersymmetry transformations and matching terms order by order in Ω we obtain the action of supersymmetry on the boundary fields:

$$\begin{aligned}
\delta_\eta z_1 &= i \overline{\alpha^\eta}^1 \alpha_1^\psi && (4.15)\\
\delta_\eta z_2 &= -\frac{i}{2} \overline{\beta^\eta}^2 \alpha_1^\psi + \frac{i}{1-2m} \overline{\alpha^\eta}^2 \tilde{h}^{ab} \tilde{\gamma}_a \tilde{\nabla}_b \alpha_1^\psi && (4.16)\\
\delta_\eta z_3 &= -i \overline{\alpha^\eta}^2 \beta_1^\psi && (4.17)\\
\delta_\eta z_4 &= \frac{i}{2} \overline{\beta^\eta}^1 \beta_1^\psi + \frac{i}{1+2m} \overline{\alpha^\eta}^1 \tilde{h}^{ab} \tilde{\gamma}_a \tilde{\nabla}_b \beta_1^\psi && (4.18)\\
\delta_\eta z_1^\dagger &= -i \overline{\alpha^\eta}^2 \alpha_2^\psi && (4.19)\\
\delta_\eta z_2^\dagger &= -\frac{i}{2} \overline{\beta^\eta}^1 \alpha_2^\psi - \frac{i}{1-2m} \overline{\alpha^\eta}^1 \tilde{h}^{ab} \tilde{\gamma}_a \tilde{\nabla}_b \alpha_2^\psi && (4.20)\\
\delta_\eta z_3^\dagger &= -i \overline{\alpha^\eta}^1 \beta_2^\psi && (4.21)\\
\delta_\eta z_4^\dagger &= -\frac{i}{2} \overline{\beta^\eta}^2 \beta_2^\psi + \frac{i}{1+2m} \overline{\alpha^\eta}^2 \tilde{h}^{ab} \tilde{\gamma}_a \tilde{\nabla}_b \beta_2^\psi && (4.22)\\
\delta_\eta \alpha_1^\psi &= 2\left(m - \frac{3}{2}\right) z_1 \beta_1^\eta + 2(1-2m) z_2 \alpha_2^\eta + 2\tilde{h}^{ab} \tilde{\gamma}_a \tilde{\nabla}_b z_1 \alpha_1^\eta && (4.23)\\
\delta_\eta \alpha_2^\psi &= 2\left(m - \frac{3}{2}\right) z_1^\dagger \beta_2^\eta - 2(1-2m) z_2^\dagger \alpha_1^\eta - 2\tilde{h}^{ab} \tilde{\gamma}_a \tilde{\nabla}_b z_1^\dagger \alpha_2^\eta && (4.24)\\
\delta_\eta \beta_1^\psi &= -2\left(m + \frac{3}{2}\right) z_3 \beta_2^\eta + 2(1+2m) z_4 \alpha_1^\eta - 2\tilde{h}^{ab} \tilde{\gamma}_a \tilde{\nabla}_b z_3 \alpha_2^\eta && (4.25)\\
\delta_\eta \beta_2^\psi &= 2\left(m + \frac{3}{2}\right) z_3^\dagger \beta_1^\eta + 2(1+2m) z_4^\dagger \alpha_2^\eta - 2\tilde{h}^{ab} \tilde{\gamma}_a \tilde{\nabla}_b z_3^\dagger \alpha_1^\eta \,, && (4.26)
\end{aligned}$$

where we have defined the complex boundary scalars

$$\begin{aligned}
z_1 &= \frac{1}{2}(\alpha_1 + i\alpha_2) \,, \quad z_1^\dagger = \frac{1}{2}(\alpha_1 - i\alpha_2) && (4.27)\\
z_2 &= \frac{1}{2}(\beta_3 + i\beta_4) \,, \quad z_2^\dagger = \frac{1}{2}(\beta_3 - i\beta_4) && (4.28)\\
z_3 &= \frac{1}{2}(\alpha_3 + i\alpha_4) \,, \quad z_3^\dagger = \frac{1}{2}(\alpha_3 - i\alpha_4) && (4.29)\\
z_4 &= \frac{1}{2}(\beta_1 + i\beta_2) \,, \quad z_4^\dagger = \frac{1}{2}(\beta_1 - i\beta_2) \,. && (4.30)
\end{aligned}$$

We now consider a subspace of the set of Killing spinors η associated with some choice of Poincaré coordinates

$$ds^2 = \frac{1}{z^2}\left(-dt^2 + dz^2 + dx_1^2 + dx_2^2 + dx_3^2\right) \,, \quad z \geq 0 \,. \tag{4.31}$$

Solutions to the conformal Killing spinor equation are then $\eta_i = \Omega^{-1/2}\varepsilon_i$, where ε_i are constant spinors satisfying $\tilde{P}_-\varepsilon_1 = 0, \tilde{P}_+\varepsilon_2 = 0$.

The 4-component bulk spinors $\alpha_i^\psi, \beta_i^\psi, \varepsilon_i$ satisfy projection conditions defined by $\tilde{P}_\pm$. As a result, they define 2-component spinors living on the boundary $\mathbb{R}^4$. To make this explicit, let the index $\bar{j}$ run over $t, \vec{x}$ and take the four-dimensional Dirac matrices to be

$$\gamma^{\bar{j}} = (\gamma^{\hat{0}}, \gamma^{\hat{k}}) = \begin{pmatrix} 0 & -i\sigma^{\bar{j}} \\ -i\bar{\sigma}^{\bar{j}} & 0 \end{pmatrix}, \tag{4.32}$$

where we have defined $\sigma^{\bar{j}} = (I_2, \vec{\sigma})$ and $\bar{\sigma}^{\bar{j}} = (I_2, -\vec{\sigma})$. Now, the radial projectors are $\tilde{P}_\pm = \frac{1}{2}(1 \pm \gamma_z)$ and it is natural to choose $\gamma_z = \gamma^{\hat{4}}$. Note however, that $\gamma^{\hat{4}} = i\gamma^{\hat{0}}\gamma^{\hat{1}}\gamma^{\hat{2}}\gamma^{\hat{3}}$ which serves as the "boundary γ_5." So, the radial projectors match onto chiral projectors on the boundary, and then a four-component bulk spinor in five dimensions that has been acted on with $\tilde{P}_\pm$ gets mapped to a two-component left or right-handed Weyl spinor in four dimensions. In particular, we have

$$\alpha_1^\psi = \begin{pmatrix} \alpha_\kappa \\ 0 \end{pmatrix}, \quad \alpha_2^\psi = \begin{pmatrix} 0 \\ i\alpha^{\dagger\dot{\kappa}} \end{pmatrix}, \quad \beta_1^\psi = \begin{pmatrix} 0 \\ \beta^{\dagger\dot{\kappa}} \end{pmatrix}, \quad \beta_2^\psi = \begin{pmatrix} -i\beta_\kappa \\ 0 \end{pmatrix} \tag{4.33}$$

$$\varepsilon_1 = \begin{pmatrix} 0 \\ \varepsilon^{\dagger\dot{\kappa}} \end{pmatrix}, \quad \varepsilon_2 = \begin{pmatrix} -i\varepsilon_\kappa \\ 0 \end{pmatrix}. \tag{4.34}$$

Our conventions for two-component spinors are the same as in the previous section. We have further introduced conjugate spinors and dotted spinor indices, $(\chi_\kappa)^\dagger = \chi^\dagger_{\dot{\kappa}}$. Dotted indices are raised and lowered by contracting with the second index of $\epsilon^{\dot{\kappa}\dot{\lambda}}, \epsilon_{\dot{\kappa}\dot{\lambda}}$, with $\epsilon^{\dot{1}\dot{2}} = 1 = -\epsilon_{\dot{1}\dot{2}}$. We also note the index placement $\sigma^{\bar{j}} = \sigma^{\bar{j}}_{\kappa\dot{\lambda}}$ and $\bar{\sigma}^{\bar{j}} = \bar{\sigma}^{\bar{j}\dot{\kappa}\lambda}$. The calculations below use the identity $\psi^\dagger\bar{\sigma}^{\bar{j}}\chi = -\chi\sigma^{\bar{j}}\psi^\dagger$.

In this notation, the Poincaré supersymmetries defined by $\eta_i = \Omega^{-1/2}\varepsilon_i$ may be written

$$\delta_\varepsilon z_1 = \varepsilon\alpha \tag{4.35}$$
$$(1-2m)\delta_\varepsilon z_2 = \varepsilon^\dagger\bar{\sigma}^{\bar{j}}\partial_{\bar{j}}\alpha \tag{4.36}$$
$$\delta_\varepsilon \alpha = -2i(1-2m)z_2\varepsilon - 2i\sigma^{\bar{j}}\partial_{\bar{j}}z_1\varepsilon^\dagger \tag{4.37}$$
$$\delta_\varepsilon z_3^\dagger = i\varepsilon\beta \tag{4.38}$$
$$(1+2m)\delta_\varepsilon z_4^\dagger = -i\varepsilon^\dagger\bar{\sigma}^{\bar{j}}\partial_{\bar{j}}\beta \tag{4.39}$$
$$\delta_\varepsilon \beta = 2(1+2m)z_4^\dagger\varepsilon - 2\sigma^{\bar{j}}\partial_{\bar{j}}z_3^\dagger\varepsilon^\dagger. \tag{4.40}$$

The transformations of the complex conjugate fields may be obtained by taking the complex conjugate of the above relations.

Under (4.35)-(4.40), boundary fields mix only within each of the disjoint sets (z_1, z_2, α), $(z_3^\dagger, z_4^\dagger, \beta)$. We may therefore construct useful boundary superfields from each set separately,

$$\Phi_1 = z_1 + \theta\alpha - i(1-2m)\theta\theta z_2 \qquad \Phi_2 = z_3^\dagger + i\theta\beta + i(1+2m)\theta\theta z_4^\dagger. \tag{4.41}$$

Taking the conformal dimension of θ to be $-1/2$, we note that the superfield $\Phi_{1,2}$ has conformal dimension $\frac{3}{2} \mp m$. One may now check that (4.35-4.40) can be written as a superspace covariant derivative acting on superfields,

$$\delta_\varepsilon \Phi = \left(\varepsilon^\kappa \frac{\partial}{\partial\theta^\kappa} - 2i\theta\sigma^{\bar{j}}\varepsilon^\dagger\partial_{\bar{j}}\right)\Phi. \tag{4.42}$$

Similarly one can define the conjugate superfields

$$\begin{gathered}\Phi_1^\dagger = z_1^\dagger + \theta^\dagger\alpha^\dagger + i(1-2m)\theta^\dagger\theta^\dagger z_2^\dagger, \qquad \Phi_2^\dagger = z_3 - i\theta^\dagger\beta^\dagger - i(1+2m)\theta^\dagger\theta^\dagger z_4, \\ \text{and} \quad \delta_\varepsilon\Phi^\dagger = \left(\varepsilon^\dagger_{\dot{\kappa}}\frac{\partial}{\partial\theta^\dagger_{\dot{\kappa}}} - 2i\theta^\dagger\bar{\sigma}^{\bar{j}}\varepsilon\partial_{\bar{j}}\right)\Phi^\dagger.\end{gathered} \tag{4.43}$$

Finally, using the above relations the total flux can be expressed as

$$\mathcal{F} = \left[-i\int_{\mathcal{I}} d^4S \int d^2\theta\,(\delta_1\Phi_1\delta_2\Phi_2 - \delta_1\Phi_2\delta_2\Phi_1)\right] + \left[\quad\right]^\dagger. \tag{4.44}$$

It is now clear that for any function W, the boundary condition

$$\Phi_1 = i\,\frac{\delta W(\Phi_2)}{\delta\Phi_2} \tag{4.45}$$

conserves the inner product and is invariant under the Poincaré supersymmetries.

In terms of the original component fields we have

$$z_1 = iW'(z_3^\dagger), \quad (1-2m)z_2 = -i(1+2m)W''(z_3^\dagger)z_4^\dagger + \frac{1}{4}W'''(z_3^\dagger)\overline{\beta^\psi}^{\,1}\beta_2^\psi, \quad \alpha_1^\psi = -iW''(z_3^\dagger)\beta_2^\psi\,. \tag{4.46}$$

For general W these boundary conditions break the conformal (and thus superconformal) symmetry; however, for the special choice

$$W(\Phi_2) = \frac{3/2+m}{3}\,q\,(\Phi_2)^{\frac{3}{3/2+m}} \tag{4.47}$$

the full $O(4,2)$ symmetry is preserved and so is the full supersymmetry defined by (4.15-4.26) for the arbitrary Killing spinors η_i. I.e., superconformal symmetry is preserved on the boundary.

As a simple example, consider the linear boundary condition $\Phi_1 = iq\Phi_2$. In terms of the component boundary fields this is

$$\alpha_1 = q\alpha_4\,, \quad \alpha_2 = q\alpha_3\,, \quad \beta_4 = -q\,\frac{1+2m}{1-2m}\beta_1\,, \quad \beta_3 = -q\,\frac{1+2m}{1-2m}\beta_2 \tag{4.48}$$

$$\alpha_1^\psi = -iq\beta_2^\psi\,, \quad \alpha_2^\psi = -iq\beta_1^\psi\,. \tag{4.49}$$

Note that these boundary conditions relate the two scalar fields to each other ($\alpha_{1,2}$ to $\alpha_{3,4}$), ($\beta_{1,2}$ to $\beta_{3,4}$). As in the AdS$_4$ theory treated above, boundary conditions relating α_I to β_I cannot be supersymmetrized to respect the Poincaré supersymmetry on the boundary.

V. $\mathcal{N} = (1,0)$ SUPERSYMMETRY IN $d = 3$

In $2+1$ dimensions, the AdS supergroup has the factored form $G_L \times G_R$, and so AdS_3 supergravity theories can be labeled as having $\mathcal{N} = (p,q)$ supersymmetry [33]. In such theories, a Majorana (real) spinor has one degree of freedom on-shell, and so should correspond via supersymmetry to one real scalar. As noted above however, in odd dimensions there does not seem to be a Lorentz invariant way of imposing generalized boundary conditions on a single fermion without introducing derivatives. So let us begin with a theory containing two copies of a scalar multiplet. This theory in fact has $\mathcal{N} = (2,0)$ supersymmetry, but we will only find interesting boundary conditions that preserve a $(1,0)$ subalgebra. The theory [34, 35, 36] consists of two scalars $\phi_i = (\phi_1,\phi_2)$ and two Majorana fermions $\hat\psi^A = (\hat\psi^1,\hat\psi^2)$ with Lagrangian (2.1), where $m_\phi \equiv m_{\phi_1} = m_{\phi_2}$ and $m \equiv m_{\psi_1} = m_{\psi_2}$. To match the usual normalization of the action for Majorana fermions we define $\psi^A \equiv \sqrt{2}\,\hat\psi^A$ and work exclusively with this rescaled spinor. The fermions ψ^A obey the Majorana condition $\overline{\psi} = \psi^T C$, where $C = i\sigma^2$ is the charge conjugation matrix. With the real gamma matrix representation $\gamma^{\hat a} = (i\sigma^2,\sigma^1,\sigma^3)$, the Majorana condition amounts to the reality condition $\psi = \psi^*$. Other conventions for 2-component spinors are the same as given above.

For

$$m_\phi^2 = m^2 + m - \frac{3}{4}\,, \tag{5.1}$$

the action is invariant under the $\mathcal{N} = (2,0)$ supersymmetry transformations

$$\delta_\eta\phi_i \;=\; \frac{i}{2}\overline{\eta}\Gamma_i\psi \tag{5.2}$$

$$\delta_\eta\psi \;=\; -\frac{i}{\sqrt{2}}\left[\gamma^a\nabla_a\phi_i\Gamma_i\eta + \left(m-\frac{1}{2}\right)\phi_i\Gamma_i\eta\right]. \tag{5.3}$$

Here we have suppressed the spinor labels $A,B,\ldots = 1,2$ and the matrices Γ_i^{AB} are $\Gamma_1 = \sigma^1, \Gamma_2 = \sigma^3$. The supersymmetry-generating parameters η^A are Majorana Killing spinors satisfying

$$\nabla_a\eta^A + \frac{1}{2}\gamma_a\eta^A = 0\,. \tag{5.4}$$

As in the higher dimensional cases treated above, massless fermions correspond to conformally coupled scalar fields, $m_\phi^2 = -3/4$, and the scalars always satisfy the BF bound, $m_\phi^2 \geq m_{BF}^2 = -1$. The BF bound is saturated at $m = -1/2$, and m_ϕ^2 reaches $m_{BF}^2 + 1$ at $m = 1/2$. We once again restrict attention to the range $-1/2 < m < 1/2$. The theory studied in [34] is a limiting case of the theory in [35], corresponding to choosing $m = 1/2$.

Solutions to the Killing spinor equation have leading terms as $\Omega \to 0$ given by

$$\eta^A = \eta_+^A \Omega^{-1/2} + \frac{1}{2}\eta_-^A \Omega^{1/2} + \dots\,, \tag{5.5}$$

where $\eta_\pm^A = \tilde{P}_\pm U^A$ and U^A is a constant spinor. We expect that valid boundary conditions will relate α_ψ^1 to α_ψ^2 and β_ψ^1 to β_ψ^2, since these spinors exhibit the same properties under the radial projectors.

Inserting the asymptotic expansions of ϕ_i, ψ^A, η^A into the supersymmetry transformations and matching terms order by order in Ω gives the action of supersymmetry on our boundary fields [36]:

$$\begin{aligned}
\delta_\eta \alpha_i &= \frac{i}{\sqrt{2}}\,\overline{\eta_+}\,\Gamma_i \alpha_\psi && (5.6)\\
\delta_\eta \beta_i &= \frac{i}{2\sqrt{2}}\,\overline{\eta_-}\,\Gamma_i \beta_\psi + \frac{1}{1+2m}\frac{i}{\sqrt{2}}\,\overline{\eta_+}\,\Gamma_i \tilde{h}^{ab}\tilde{\gamma}_a \tilde{\nabla}_b \beta_\psi && (5.7)\\
\delta_\eta \alpha_\psi &= -\frac{1}{\sqrt{2}}\left[\left(m - \frac{1}{2}\right)\alpha_i \Gamma_i \eta_- + \tilde{h}^{ab}\tilde{\gamma}_a \tilde{\nabla}_b \alpha_i \Gamma_i \eta_+\right] && (5.8)\\
\delta_\eta \beta_\psi &= -\frac{1}{\sqrt{2}}(1+2m)\beta_i \Gamma_i \eta_+\,. && (5.9)
\end{aligned}$$

Here we shall only attempt to preserve the $(1,0)$ supersymmetry transformations[6] given by setting $\eta^1 = 0, \eta^2 \equiv \eta$. We then have

$$\begin{aligned}
\delta_\eta \alpha_1 &= \frac{i}{\sqrt{2}}\,\overline{\eta_+}\,\alpha_\psi^1 && (5.10)\\
\delta_\eta \alpha_2 &= -\frac{i}{\sqrt{2}}\,\overline{\eta_+}\,\alpha_\psi^2 && (5.11)\\
\delta_\eta \beta_1 &= \frac{i}{2\sqrt{2}}\,\overline{\eta_-}\,\beta_\psi^1 + \frac{1}{1+2m}\frac{i}{\sqrt{2}}\,\overline{\eta_+}\,\tilde{h}^{ab}\tilde{\gamma}_a \tilde{\nabla}_b \beta_\psi^1 && (5.12)\\
\delta_\eta \beta_2 &= -\frac{i}{2\sqrt{2}}\,\overline{\eta_-}\,\beta_\psi^2 - \frac{1}{1+2m}\frac{i}{\sqrt{2}}\,\overline{\eta_+}\,\tilde{h}^{ab}\tilde{\gamma}_a \tilde{\nabla}_b \beta_\psi^2 && (5.13)\\
\delta_\eta \alpha_\psi^1 &= -\frac{1}{\sqrt{2}}\left[\left(m - \frac{1}{2}\right)\alpha_1 \eta_- + \tilde{h}^{ab}\tilde{\gamma}_a \tilde{\nabla}_b \alpha_1 \eta_+\right] && (5.14)\\
\delta_\eta \alpha_\psi^2 &= \frac{1}{\sqrt{2}}\left[\left(m - \frac{1}{2}\right)\alpha_2 \eta_- + \tilde{h}^{ab}\tilde{\gamma}_a \tilde{\nabla}_b \alpha_2 \eta_+\right] && (5.15)\\
\delta_\eta \beta_\psi^1 &= -\frac{1}{\sqrt{2}}(1+2m)\beta_1 \eta_+ && (5.16)\\
\delta_\eta \beta_\psi^2 &= \frac{1}{\sqrt{2}}(1+2m)\beta_2 \eta_+\,. && (5.17)
\end{aligned}$$

We now consider a subspace of the set of Killing spinors η associated with some choice of Poincaré coordinates

$$ds^2 = \frac{1}{z^2}\left(-dt^2 + dz^2 + dx_1^2\right)\,, \quad z \geq 0\,. \tag{5.18}$$

Solutions to the Killing spinor equation are then $\eta = \Omega^{-1/2}\varepsilon_+$, where ε_+ is a constant spinor satisfying $\tilde{P}_-\varepsilon_+ = 0$.

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[6] The full $(2,0)$ transformations can be preserved with Dirichlet or Neumann type boundary conditions [36]; whether this can be done with more general boundary conditions as well is a matter for further investigation. It is not immediately obvious how to do so, since the $(2,0)$ boundary supersymmetry multiplets are $(\alpha_i, \alpha_\psi^A)$, (β_i, β_ψ^A), and thus the natural boundary conditions for the spinors would lead to relating fields in the same multiplet.

Let the index μ run over t, x_1 and take the two-dimensional Dirac matrices to be $\gamma^\mu = (\gamma^{\hat{0}}, \gamma^{\hat{1}}) = (i\sigma^2, \sigma^1)$. Now, the radial projectors are $\tilde{P}_\pm = \frac{1}{2}(1 \pm \gamma_z)$ and it is natural to choose $\gamma_z = \gamma^{\hat{2}}$. Note however, that $\gamma^{\hat{2}} = \gamma^{\hat{0}}\gamma^{\hat{1}} = \sigma^3$, which serves as the "boundary γ_5." So, the radial projectors match onto chiral projectors on the boundary, and then a three-dimensional bulk spinor that has been acted on with $\tilde{P}_\pm$ gets mapped to a Weyl spinor in two dimensions. Note that for any two-component spinors $\chi_\pm \equiv \tilde{P}_\pm \chi, \psi_\pm \equiv \tilde{P}_\pm \psi$, we have $\chi_\pm \psi_\pm = 0$.

Under (5.10)-(5.17), boundary fields mix only within each of the disjoint sets $(\alpha_1, \alpha_\psi^1)$, $(\alpha_2, \alpha_\psi^2)$, (β_1, β_ψ^1), and (β_2, β_ψ^2). This suggests that we define the scalar boundary superfields

$$\Phi_1 = \alpha_1 + \overline{\theta_+}\alpha_\psi^1, \quad \Phi_2 = \alpha_2 - \overline{\theta_+}\alpha_\psi^2 \tag{5.19}$$

and the spinor boundary superfields

$$\Psi_1 = \beta_\psi^2 - i(1+2m)\theta_+\beta_2, \quad \Psi_2 = \beta_\psi^1 + i(1+2m)\theta_+\beta_1\,. \tag{5.20}$$

We again take θ_+ to have conformal dimension $-1/2$ so that the scalar superfields both have conformal dimension $\frac{1}{2} - m$, while the spinor superfields both have conformal dimension $1 + m$. One may now check that (5.10-5.17) (with $\eta_- \to 0, \eta_+ \to \varepsilon_+$) can be written as a superspace covariant derivative acting on superfields,

$$\delta_\varepsilon \Phi = \frac{1}{\sqrt{2}}\left(i\overline{\varepsilon_+}\frac{\partial}{\partial\overline{\theta_+}} + \overline{\varepsilon_+}\gamma^\mu\theta_+\partial_\mu\right)\Phi, \quad \delta_\varepsilon \Psi = \frac{1}{\sqrt{2}}\left(i\varepsilon_+\frac{\partial}{\partial\theta_+} + \overline{\varepsilon_+}\gamma^\mu\theta_+\partial_\mu\right)\Psi\,. \tag{5.21}$$

Using relations (5.19) and (5.20), the total flux can be expressed as

$$\mathcal{F} = \left[-i\int_{\mathcal{I}} d^2S \int d\theta_+ \left(\delta_1\Phi_1\delta_2\Psi_2 - \delta_1\Phi_2\delta_2\Psi_1\right)\right] - \left[\delta_1 \leftrightarrow \delta_2\right]. \tag{5.22}$$

It is now clear that for any function f, the boundary condition

$$\Phi_1 = f(\Phi_2)\,, \qquad \Psi_1 = \frac{\delta f(\Phi_2)}{\delta\Phi_2}\Psi_2 \tag{5.23}$$

conserves the inner product and is invariant under the Poincaré supersymmetries. These boundary conditions may be summarized through the spinor potential $\widetilde{W} = f(\Phi_2)\Psi_2$, in terms of which the deformation of any dual CFT action is $-i\int d^2S \int d\theta_+ \widetilde{W}$.

In terms of the component fields we have

$$\begin{aligned} \alpha_1 &= f(\alpha_2) && (5.24)\\ (1+2m)\beta_2 &= -(1+2m)f'(\alpha_2)\beta_1 + if''(\alpha_2)\overline{\beta_\psi^1}\alpha_\psi^2 && (5.25)\\ \alpha_\psi^1 &= -f'(\alpha_2)\alpha_\psi^2 && (5.26)\\ \beta_\psi^2 &= f'(\alpha_2)\beta_\psi^1\,. && (5.27) \end{aligned}$$

For general $\widetilde{W}$, these boundary conditions break the conformal (and thus superconformal) symmetry. However, for the special choice of linear boundary conditions $\Phi_1 = q\Phi_2\,, \Psi_1 = q\Psi_2\,$, that is,

$$\alpha_1 = q\alpha_2\,, \quad \beta_2 = -q\beta_1\,, \quad \alpha_\psi^1 = -q\alpha_\psi^2\,, \quad \beta_\psi^2 = q\beta_\psi^1\,, \tag{5.28}$$

the full AdS symmetry is preserved and so is the full supersymmetry defined by (5.10-5.17) for the arbitrary Killing spinors η. For this case, $\widetilde{W}$ has conformal dimension 3/2 and provides a marginal deformation of the dual CFT. The result that the linear boundary conditions preserve superconformal symmetry for any $|m| < 1/2$ is associated with the two scalars always having equal masses; in the $d = 4, 5$ cases considered above, the scalars masses coincide only when $m = 0$.

Alternatively, one can define two scalar superfields

$$\Xi_1 = \alpha_1 + \overline{\theta_+}\alpha_\psi^1 + i\overline{\theta_-}\beta_\psi^2 + (1+2m)\overline{\theta_-}\theta_+\beta_2, \quad \Xi_2 = \alpha_2 - \overline{\theta_+}\alpha_\psi^2 + i\overline{\theta_-}\beta_\psi^1 - (1+2m)\overline{\theta_-}\theta_+\beta_1 \tag{5.29}$$

of conformal dimension $\frac{1}{2} - m$ (where we take θ_- to have conformal dimension $-\frac{1}{2} - 2m$) and obtain the supersymmetry transformations by acting with the same superspace derivative given in (5.21). In terms of these superfields, the flux can be expressed as

$$\mathcal{F} = \int_{\mathcal{I}} d^2S \int d\theta_+ d\overline{\theta_-}\left(\delta_1\Xi_2\delta_2\Xi_1 - \delta_1\Xi_1\delta_2\Xi_2\right)\,. \tag{5.30}$$

d	m_ψ	# Traces
3	any value between $-\frac{1}{2}$ and $\frac{1}{2}$	2
4	0	2
	$\pm 1/3$	3
5	0	2

TABLE II: Cases with integer number of traces and superconformal symmetry. Note that in $d = 4$ such cases arise for both double-trace deformations ($m_\psi = 0$) and triple-trace deformations ($m_\psi = \pm 1/3$).

The boundary condition $\Xi_1 = W'(\Xi_2)$ conserves the inner product and, when written out in terms of the component fields, gives the same expressions (5.24-5.27) for $f = W'$ or $\widetilde{W} = -i \int d\overline{\theta_-}\, W$.

If instead we had set $\eta^1 = \eta, \eta^2 = 0$, we would have obtained a different set of $\mathcal{N} = (1,0)$ supersymmetry transformations, which can be obtained from those given above by replacing $\alpha^1_\psi \to \alpha^2_\psi\,, \alpha^2_\psi \to -\alpha^1_\psi\,, \beta^1_\psi \to \beta^2_\psi\,, \beta^2_\psi \to -\beta^1_\psi$. Performing a similar analysis in this case leads to the general boundary conditions

$$\alpha_1 = W'(\alpha_2), \quad (1+2m)\beta_2 = -(1+2m)W''(\alpha_2)\beta_1 - iW'''(\alpha_2)\overline{\beta^2_\psi}\alpha^1_\psi, \tag{5.31}$$

$$\alpha^2_\psi = W''(\alpha_2)\alpha^1_\psi, \quad \beta^1_\psi = -W''(\alpha_2)\beta^2_\psi\,. \tag{5.32}$$

VI. DISCUSSION

Our study began with a general analysis of boundary conditions, consistent with finiteness and conservation of the standard inner product, for Dirac fermions in AdS spacetime (as had been previously done for bosonic fields [7, 8] and for fermions in AdS$_4$ [7]). For any real mass and any $d \geq 2$, one may choose boundary conditions that make our fermions stable at the level of linear perturbations; formally, the condition $(m_\psi \pm \frac{1}{2})^2 \geq 0$ is analogous to the Breitenlohner-Freedman bound [7] for scalars $m^2_\phi \geq -(d-1)^2/4$. For $m^2_\psi \geq 1/4$, only the faster falloff mode is normalizeable, so boundary conditions must fix the coefficient α_ψ of the slow falloff mode. For $0 \leq m^2_\psi < 1/4$, all modes are normalizeable and more general boundary conditions are allowed. This is directly analogous to the situation for scalars, where general boundary conditions are permitted in the range $m^2_{BF} \leq m^2_\phi < m^2_{BF} + 1$. However, for d odd, the only Lorentz-invariant derivative-free boundary conditions for a theory with a single fermion are $\alpha_\psi = 0$ or $\beta_\psi = 0$.

For the examples of supersymmetry studied here, fermion and scalar masses (m_ψ, m_ϕ) are related in all dimensions by

$$m^2_{\phi,\pm}(m) = m^2_\psi \pm \frac{m_\psi}{\ell} - \frac{d(d-2)}{4\ell^2}\,, \tag{6.1}$$

where we have restored factors of the AdS radius ℓ and the $\pm$ denotes the fact that two scalar masses are typically allowed for a given fermion mass. This formula also holds in $d = 2$ [14]. Our results for fermion boundary conditions at mass m_ψ typically agree with those for scalars at mass $m_{\phi,\pm}$ when m_ψ and m_ϕ satisfy (6.1). The one exception occurs for $m_\psi = \mp 1/(2\ell)$, which implies $m^2_{\phi,\pm} = m^2_{BF}$ but $m^2_{\phi,\mp} = m^2_{BF} + 1/\ell^2$. Since the slow fall-off scalar mode is normalizeable for $m^2_{\phi,\pm}$, but not for $m^2_{\phi,\mp}$, it is clear that the fermion cannot agree with both scalars. In fact, the slow fall-off fermion modes fail to be normalizeable in the standard inner product[7]. Thus, there are no supersymmetric multi-trace boundary conditions when the BF bound is saturated. This is consistent with *i*) the results of [22], which found that for a single scalar at the BF bound, the Witten-Nester proof of the positive energy theorem does not apply unless one turns off the logarithmic mode and *ii*) the results of [37], which argued (in the context of maximal gauged supergravity on AdS$_5$) that turning on the logarithmic branch leads to energies unbounded below.

We used such results to classify boundary conditions which preserve supersymmetry (either a so-called Poincaré superalgebra involving half of the supercharges or the full superalgebra) for certain choices of field content. In general, linear boundary conditions can preserve only the Poincaré subalgebra of supercharges. The same is true of

[7] Though it might be interesting to reexamine this issue using the techniques of [10].

boundary conditions which would correspond to deformations of a dual field theory involving an integer number of traces. Exceptions occur for special values of the fermion masses, and for $d = 3$ due to the nature of the AdS_3 chiral supermultiplet. These exceptions are summarized in Table II. For $d = 4, m = 0$, our results reduce to those of [11] and yield the boundary conditions of [7] in a suitable limit.

It may be interesting to perform a similar analysis including the effects of backreaction, to investigate general boundary conditions for vector and graviton supermultiplets, or to consider extended supersymmetry. However, of most interest would be a comparison with a classification of supersymmetric deformations of a dual field theory. We close by discussing the details of 10-dimensional IIB supergravity on $AdS_5 \times S^5$, 11-dimensional supergravity on $AdS_4 \times S^7$, and 10-dimensional IIA supergravity on $AdS_4 \times \mathbb{CP}^3$. We then draw conclusions for the corresponding dual theories; i.e., for $\mathcal{N} = 4$ super Yang-Mills in 3+1 dimensions [1] and for the theories described in [25, 26].

The case of $AdS_5 \times S^5$ can be dealt with quickly. From [38], we see that after Kaluza-Klein reduction on the S^5, all spin 1/2 fields have AdS_5 masses (in our notation) with magnitude greater or equal to 1/2. There are no allowed deformations of boundary conditions for spin-1/2 fields, and thus no supersymmetric deformations of boundary conditions of the type discussed here. While we have not studied the spin-3/2 fields, for $d = 5$ one does not expect to be able to deform boundary conditions associated with either vector or tensor fields in a Lorentz-invariant manner without introducing ghosts [8, 9, 10]. As a result, deformations of the spin-3/2 boundary conditions are unlikely to be allowed, and supersymmetric deformations will certainly be forbidden. We conclude that there are no (relevant or marginal) multi-trace deformations of the dual $\mathcal{N} = 4$ super Yang-Mills theory which preserve even $\mathcal{N} = 1$ Poincaré supersymmetry on the boundary.

Let us now consider $AdS_4 \times S^7$ and $AdS_4 \times \mathbb{CP}^3$. From [39] and [40] we see that these theories do contain fermions with masses $|m| < 1/2$. Let us discuss the S^7 case for definiteness, though the $\mathbb{CP}^3$ case is similar. For $AdS_4 \times S^7$, there is a single SO(8) multiplet of spin-1/2 fields in the desired mass range: the 56_s representation of SO(8) with mass $m = 0$. In addition, there are two SO(8) multiplets (35_v and 35_c) of conformally coupled scalars. Choosing an $\mathcal{N} = 1$ super-Poincaré algebra on the boundary, we may assemble from these fields 35 pairs of boundary superfields $\Phi_\pm$ as described in section III. Allowed deformations with integer numbers of traces are characterized by polynomials in the 35 Φ_+. Since each superfield has conformal dimension 1, there are no relevant deformations and the marginal deformations are labeled by the $\binom{35}{2} = 595$ quadratic monomials formed from these fields. I.e., there is a 595-dimensional manifold of conformal theories connected by double-trace deformations. Since our superfields do not form a well-defined SO(8) representation, none of these deformations will preserve SO(8) symmetry (as is expected since we singled out an $\mathcal{N} = 1$ subalgebra of the full supersymmetries). Analogous reasoning leads to a (somewhat smaller) manifold of conformal theories dual to $AdS_4 \times \mathbb{CP}^3$. We reached this conclusion ignoring all bulk interactions, but from e.g. [21, 22] and the fact that the theory has a symmetry that changes the sign of the relevant scalars, one can see that including interactions will not change this analysis. The symmetry implies that the potential is even and forbids certain logarithms that might otherwise be problematic for $m_\phi^2 = -2$.

Strictly speaking, the above conclusions hold only at large N and one should ask if our marginal deformations might become relevant or irrelevant at finite N. While we cannot rule this out, from the analysis of [41] it is clear that our deformations remain marginal when leading $1/N$ corrections are included. Indeed, the manifest AdS isometry appears to control bulk perturbation theory in G_N to all orders, so that our deformations remain exactly marginal at all orders in the $1/N$ expansion.

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APPENDIX A: THE DIRAC EQUATION IN ANTI-DE SITTER SPACETIME

In this appendix, we reduce the Dirac equation in AdS_d to a set of coupled, first order differential equations which may then be decoupled and easily solved. The Dirac equation in static, spherically symmetric spacetimes has been studied in e.g., [24, 42, 43], and in particular we now review the results of [24].

The metric for AdS_d takes the form

$$ds^2 = -h(r)dt^2 + h^{-1}(r)dr^2 + r^2 d\Omega_{d-2}^2 \,, \tag{A1}$$

where $h(r) = 1 + r^2$. Following [24], we define the "Cartesian" coordinates

$$x^1 = r\cos\theta_1 \sin\theta_2 \ldots \sin\theta_{d-2}, \quad x^2 = r\sin\theta_1 \sin\theta_2 \ldots \sin\theta_{d-2}, \quad \ldots, \quad x^{d-1} = r\cos\theta_{d-2}$$

$$\text{with} \qquad r^2 = \textstyle\sum_{k=1}^{d-1}(x^k)^2 \equiv x^k x^k \,. \tag{A2}$$

Then the spatial part of the metric (A1) can be written as

$$g_{ij} = \delta_{ij} - \frac{1}{r^2}\left(1 - \frac{1}{h}\right) x^i x^j \,, \qquad i,j,k,\ldots = 1,2,\ldots,d-1 \,, \tag{A3}$$

which leads us to choose the orthonormal frame

$$e^{\hat{0}} = \sqrt{h}\, dt, \quad e^{\hat{k}} = dx^k - \frac{1}{r^2}\left(1 - \frac{1}{\sqrt{h}}\right) x^k x^j dx^j \,. \tag{A4}$$

As noted in [43], for static, spherically symmetric spacetimes the connection term $\gamma^a \Gamma_a$ that appears in the Dirac equation can be computed from the simple formula

$$\gamma^a \Gamma_a = \frac{1}{2\sqrt{-g}}\, \partial_a(\sqrt{-g}\,\gamma^a) \,, \tag{A5}$$

where $g = \det g_{ab}$. Using this result, the Dirac equation can be written as

$$\frac{1}{\sqrt{h}}\gamma^{\hat{0}}\partial_t \psi + \frac{\sqrt{h}}{r^2} x^k \gamma^{\hat{k}} \left[\left(1 - \frac{1}{\sqrt{h}}\right)\left(x^j \partial_j + \frac{d-2}{2}\right) + \frac{rh'}{4h}\right]\psi + \gamma^{\hat{k}}\partial_k \psi - m\psi = 0 \,. \tag{A6}$$

The next step is to define the "angular momentum" operator $L_{ij} = -i\left(x^i \partial_j - x^j \partial_i\right)$ and the Lorentz generator $S^{ij} = \frac{1}{2}\gamma^{[\hat{i}}\gamma^{\hat{j}]}$. Then, one can show that

$$\gamma^{\hat{k}}\partial_k = \frac{i}{r^2} x^k \gamma^{\hat{k}} S^{ij} L_{ij} + \frac{1}{r^2} x^k \gamma^{\hat{k}} x^j \partial_j \,, \tag{A7}$$

which, upon rescaling $\psi = r^{-\frac{d-2}{2}} h^{-\frac{1}{4}} \tilde{\psi}$ allows (A6) to be rewritten as

$$\gamma^{\hat{0}}\partial_t \tilde{\psi} - \frac{\sqrt{h}}{r^2} x^k \gamma^{\hat{k}} \left(\frac{d-2}{2} - iS^{ij}L_{ij}\right)\tilde{\psi} + \frac{h}{r^2} x^k \gamma^{\hat{k}} x^j \partial_j \tilde{\psi} - m\sqrt{h}\tilde{\psi} = 0 \,. \tag{A8}$$

Let us first suppose that $d = 2n$ is even. Then we choose an explicit gamma matrix representation

$$\gamma^{\hat{0}} = \begin{pmatrix} -iI_{2^{n-1}} & 0 \\ 0 & iI_{2^{n-1}} \end{pmatrix}, \quad \gamma^{\hat{k}} = \begin{pmatrix} 0 & -i\tau^k \\ i\tau^k & 0 \end{pmatrix}, \quad k = 1,2,\ldots,2n-1 \,, \tag{A9}$$

where τ^k are $2^{n-1} \times 2^{n-1}$ matrices satisfying $\{\tau^i, \tau^j\} = 2\delta^{ij}$ and I_n is the $n \times n$ identity matrix. In four dimensions, the τ^k are just the standard Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \,. \tag{A10}$$

In higher dimensions, the τ^k can be constructed from tensor products of Pauli matrices (see e.g. [44]), though we will not need the explicit expressions here. To proceed further, we separate variables by making an ansatz for solutions of the Dirac equation

$$\tilde{\psi}^{\pm}(t,r,\theta_i) = \begin{pmatrix} iG^{\pm}(r)\mathscr{Y}_K^{\pm}(\theta_i) \\ F^{\pm}(r)\mathscr{Y}_K^{\mp}(\theta_i) \end{pmatrix} e^{-i\omega t} \,, \tag{A11}$$

where $\mathscr{Y}_K^{\pm}$ are 2^{n-1}-component spinor spherical harmonics [44]. These spinors satisfy

$$K\mathscr{Y}_K^{\pm} = \pm(l+n-1)\mathscr{Y}_K^{\pm}, \quad \text{for} \quad K \equiv \frac{d-2}{2} - \frac{i}{2}\tau^i\tau^j L_{ij}. \tag{A12}$$

Here $l = 0,1,2,\ldots$ is the orbital angular momentum quantum number. In $d = 4$, the operator K takes the familiar form $(1 + \vec{\sigma}\cdot\vec{L})$, with $\vec{L} = -i\vec{r}\times\vec{\partial}$ the usual angular momentum operator. In addition, the spinor harmonics have the

property $\frac{1}{r}x^j\tau^j\,\mathscr{Y}_K^{\pm}=\mathscr{Y}_K^{\mp}$, and the spinors $\psi^{\pm}$ are parity eigenstates, i.e. under parity $\psi^{\pm}\to\pm(-1)^l\psi^{\pm}$. Substituting (A11) into (A8), we obtain the coupled differential equations

$$h\frac{dF^{\pm}}{dr}=-\omega G^{\pm}\mp\frac{k\sqrt{h}}{r}F^{\pm}-m\sqrt{h}G^{\pm},\quad h\frac{dG^{\pm}}{dr}=\omega F^{\pm}\pm\frac{k\sqrt{h}}{r}G^{\pm}-m\sqrt{h}F^{\pm}\,, \tag{A13}$$

where $k=l+n-1$.

Now consider odd spacetime dimension $d=2n-1$. We choose the gamma matrix representation $\gamma^{\hat{0}}=-i\tau^{2n-1}$, $\gamma^{\hat{k}}=-i\tau^{2n-1}\tau^k$, $k=1,\ldots,2n-2$ where the τ^k are the same matrices referred to above for the $d=2n$ case. We make the ansatz

$$\tilde{\psi}^{+}=(iG^{+}\mathscr{Y}_{K,1}^{+}+F^{+}\mathscr{Y}_{K,1}^{-}),\quad \tilde{\psi}^{-}=(iG^{-}\mathscr{Y}_{K,2}^{-}+F^{-}\mathscr{Y}_{K,2}^{+}) \tag{A14}$$

where $\mathscr{Y}_{K,p}^{\pm}$ $(p=1,2)$ are 2^{n-1}-component spinor spherical harmonics [44] satisfying

$$K\mathscr{Y}_{K,p}^{\pm}=\left(\frac{d-2}{2}-\frac{i}{2}\tau^i\tau^jL_{ij}\right)\mathscr{Y}_{K,p}^{\pm}=\pm\left(l+\frac{2n-3}{2}\right)\mathscr{Y}_{K,p}^{\pm}\,, \tag{A15}$$

$$\frac{1}{r}x^k\tau^k\mathscr{Y}_{K,p}^{\pm}=\mathscr{Y}_{K,p}^{\mp}\,,\quad\text{and}\quad\gamma^{\hat{0}}\mathscr{Y}_{K,p}^{\pm}=\pm i(-1)^p\,\mathscr{Y}_{K,p}^{\pm}\,. \tag{A16}$$

Under parity, we again have $\psi^{\pm}\to\pm(-1)^l\psi^{\pm}$. Using these relations in (A8), we obtain the coupled differential equations

$$h\frac{dF^{\pm}}{dr}=-\omega G^{\pm}\mp\frac{k\sqrt{h}}{r}F^{\pm}-m\sqrt{h}G^{\pm},\quad h\frac{dG^{\pm}}{dr}=\omega F^{\pm}\pm\frac{k\sqrt{h}}{r}G^{\pm}-m\sqrt{h}F^{\pm}\,, \tag{A17}$$

where $k=l+(2n-3)/2$. We observe that these equations take exactly the same form as in the even dimensional case.

To summarize, for any $d\geq 2$ we have reduced the Dirac equation to a system of coupled ODEs

$$-\frac{dF^{\pm}}{dr_*}\mp k\csc r_*F^{\pm}-m\sec r_*G^{\pm}\;=\;\omega G^{\pm} \tag{A18}$$

$$\frac{dG^{\pm}}{dr_*}\mp k\csc r_*G^{\pm}+m\sec r_*F^{\pm}\;=\;\omega F^{\pm}\,, \tag{A19}$$

where $k=l+(d-2)/2$ and we have defined a new radial coordinate $r_*=\tan^{-1}r$ whose range is $[0,\pi/2)$. In particular, the pair F^+,G^+ are coupled together, as are the pair F^-,G^-, but there is no mixing between the two pairs. To solve these equations, we must first decouple them with a clever trick, following [42]. One first expresses the system in matrix form

$$H_{\pm}\begin{pmatrix}F^{\pm}\\G^{\pm}\end{pmatrix}=\omega\begin{pmatrix}F^{\pm}\\G^{\pm}\end{pmatrix}\,,\quad\text{where}\quad H_{\pm}=\begin{pmatrix}m\sec r_* & \frac{d}{dr_*}\pm k\csc r_*\\ -\frac{d}{dr_*}\pm k\csc r_* & -m\sec r_*\end{pmatrix} \tag{A20}$$

is the "Hamiltonian." One then defines the unitary matrix $U=\begin{pmatrix}\cos\frac{r_*}{2} & \sin\frac{r_*}{2}\\ \sin\frac{r_*}{2} & -\cos\frac{r_*}{2}\end{pmatrix}$ and performs the rotation $\begin{pmatrix}\hat{F}^{\pm}\\\hat{G}^{\pm}\end{pmatrix}\equiv U\begin{pmatrix}F^{\pm}\\G^{\pm}\end{pmatrix}$. The rotated system then satisfies

$$\hat{H}_{\pm}\begin{pmatrix}\hat{F}^{\pm}\\\hat{G}^{\pm}\end{pmatrix}=\left(\omega-\frac{1}{2}\right)\begin{pmatrix}\hat{F}^{\pm}\\\hat{G}^{\pm}\end{pmatrix}\,, \tag{A21}$$

where $\hat{H}_{\pm}=\begin{pmatrix}m\mp k & -\frac{d}{dr_*}+W_{\pm}\\ \frac{d}{dr_*}+W_{\pm} & -(m\mp k)\end{pmatrix}$, $\quad W_{\pm}=m\tan r_*\pm k\cot r_*$. Acting again with $\hat{H}_{\pm}$, we have

$$\begin{pmatrix}-\frac{d^2\hat{F}^{\pm}}{dr_*^2}+\left(W_{\pm}^2-\frac{dW_{\pm}}{dr_*}+(m\mp k)^2\right)\hat{F}^{\pm}\\ -\frac{d^2\hat{G}^{\pm}}{dr_*^2}+\left(W_{\pm}^2+\frac{dW_{\pm}}{dr_*}+(m\mp k)^2\right)\hat{G}^{\pm}\end{pmatrix}=\left(\omega-\frac{1}{2}\right)^2\begin{pmatrix}\hat{F}^{\pm}\\\hat{G}^{\pm}\end{pmatrix}\,. \tag{A22}$$

Next, let us change radial coordinates again to $x = \pi/2 - r_*$. Then x has range $(0, \pi/2]$, with the conformal boundary corresponding to $x = 0$. Thus, we obtain the decoupled second order differential equations

$$-\frac{d^2\hat{F}^{\pm}}{dx^2} + \left(\frac{\nu_-^2 - 1/4}{\sin^2 x} + \frac{\sigma_\pm^2 - 1/4}{\cos^2 x}\right)\hat{F}^{\pm} = \tilde{\omega}^2 \hat{F}^{\pm} \tag{A23}$$

$$-\frac{d^2\hat{G}^{\pm}}{dx^2} + \left(\frac{\nu_+^2 - 1/4}{\sin^2 x} + \frac{\sigma_\mp^2 - 1/4}{\cos^2 x}\right)\hat{G}^{\pm} = \tilde{\omega}^2 \hat{G}^{\pm} \tag{A24}$$

where we have defined $\tilde{\omega} = \omega - 1/2$, $\nu_\pm^2 - \frac{1}{4} = m(m \pm 1)$, and $\sigma_\pm^2 - \frac{1}{4} = k(k \pm 1)$. Solutions to these differential equations are discussed in appendix B.

APPENDIX B: NORMALIZEABLE MODES FOR DIRAC FERMIONS

We now analyze normalizeability for massive Dirac fermions using the standard inner product

$$\sigma_\Sigma(\delta_1\psi, \delta_2\psi) = i\int d^{d-1}x\sqrt{g_\Sigma}\, t_a \left(\overline{\delta_1\psi}\gamma^a\delta_2\psi - \overline{\delta_2\psi}\gamma^a\delta_1\psi\right) \tag{B1}$$

between linearized solutions. Here Σ is a hypersurface defined by $t = constant$ with unit normal t^a and g_Σ is the determinant of the induced metric on Σ. We will rely heavily on the treatment of the AdS$_d$ Dirac equation in [24] (see the summary in appendix A), and in particular on equations (A23), (A24).

Assuming that $\nu_\pm, \sigma_\pm \geq 0$ (and for now $m \geq 0$), we have $\nu_- = \left|m - \frac{1}{2}\right|$, $\nu_+ = m + \frac{1}{2}$, while $\sigma_- = \left|l + \frac{d-3}{2}\right|$, $\sigma_+ = l + \frac{d-1}{2}$. Inserting the ansatz (A11) or (A14) into (B1) and assuming that the spinor spherical harmonics are properly normalized yields

$$\sigma_\Sigma(\delta_1\psi^\pm, \delta_2\psi^\pm) = -i\int_0^{\pi/2} dx\left((\delta_1 G^\pm)^*\delta_2 G^\pm + (\delta_1 F^\pm)^*\delta_2 F^\pm\right) - (\delta_1 \leftrightarrow \delta_2)\ . \tag{B2}$$

We see that requiring the inner product to be finite is the same as requiring $\delta F^\pm, \delta G^\pm$ to be square integrable on $L^2(x \in [0, \pi/2])$. Since the rotation by U to the new radial functions $\hat{F}^\pm, \hat{G}^\pm$ is unitary, this is further equivalent to square integrability of $\delta\hat{F}^\pm, \delta\hat{G}^\pm$.

It is useful to observe that the equations of motion for fermions (A23),(A24) have been put in the same general form as that used for scalar, vector, and tensor fields in [8]. Hence, we can take advantage of the analysis already performed in that reference. For even d, the $\sigma_\pm$ are non-integer, and the general solutions to (A23), (A24) are hypergeometric functions

$$\begin{aligned}\hat{F}^\pm = {} & B_1^\pm\ (\sin x)^{\nu_- + 1/2}(\cos x)^{\sigma_\pm + 1/2}\,{}_2F_1(\zeta^{\tilde{\omega}}_{\nu_-,\sigma_\pm}, \zeta^{-\tilde{\omega}}_{\nu_-,\sigma_\pm}, 1 + \sigma_\pm, \cos^2 x) \\ & + B_2^\pm(\sin x)^{\nu_- + 1/2}(\cos x)^{-\sigma_\pm + 1/2}\,{}_2F_1(\zeta^{\tilde{\omega}}_{\nu_-,-\sigma_\pm}, \zeta^{-\tilde{\omega}}_{\nu_-,-\sigma_\pm}, 1 - \sigma_\pm, \cos^2 x)\end{aligned} \tag{B3}$$

$$\begin{aligned}\hat{G}^\pm = {} & C_1^\pm\ (\sin x)^{\nu_+ + 1/2}(\cos x)^{\sigma_\mp + 1/2}\,{}_2F_1(\zeta^{\tilde{\omega}}_{\nu_+,\sigma_\mp}, \zeta^{-\tilde{\omega}}_{\nu_+,\sigma_\mp}, 1 + \sigma_\mp, \cos^2 x) \\ & + C_2^\pm(\sin x)^{\nu_+ + 1/2}(\cos x)^{-\sigma_\mp + 1/2}\,{}_2F_1(\zeta^{\tilde{\omega}}_{\nu_+,-\sigma_\mp}, \zeta^{-\tilde{\omega}}_{\nu_+,-\sigma_\mp}, 1 - \sigma_\mp, \cos^2 x)\end{aligned} \tag{B4}$$

where $\zeta^\omega_{\nu,\sigma} = \frac{\nu + \sigma + 1 + \omega}{2}$. Near the origin, $x \sim \pi/2$, we have $\hat{F}^\pm = B_2^\pm(\cos x)^{-\sigma_\pm + 1/2} + \ldots$, $\hat{G}^\pm = C_2^\pm(\cos x)^{-\sigma_\mp + 1/2} + \ldots$. Thus $\hat{F}^\pm, \hat{G}^\pm$ are not square integrable near the origin if $\sigma_\pm \geq 1$. This inequality is always satisfied for even d, except for the cases $d = 2$ $(\sigma_\pm = 1/2)$ and $d = 4, l = 0$ $(\sigma_- = 1/2)$.

For odd d, the $\sigma_\pm$ are integers, and the second linearly independent solution to $\hat{F}^\pm, \hat{G}^\pm$ is modified (see [8]). For $\sigma_\pm \neq 0$, the solutions behave near the origin as $\hat{F}^\pm \propto B_2^\pm(\cos x)^{-\sigma_\pm + 1/2}\left((\cos x)^{-2\sigma_\pm} + \ldots\right)$, $\hat{G}^\pm \propto C_2^\pm(\cos x)^{-\sigma_\mp + 1/2}\left((\cos x)^{-2\sigma_\mp} + \ldots\right)$. Here $\sigma_\pm \geq 1$, and so $\hat{F}^\pm, \hat{G}^\pm$ are not square integrable near the origin. When $d = 3, l = 0$ we have $\sigma_- = 0$ and near the origin $\hat{F}^- = B_2^\pm(\cos x)^{1/2}\log(\cos^2 x) + \ldots$, $\hat{G}^+ = C_2^\pm(\cos x)^{1/2}\log(\cos^2 x) + \ldots$. These solutions are square integrable near $x \sim \pi/2$.

We have seen that square integrability requires $B_2^\pm = 0 = C_2^\pm$, except in the cases $\sigma_\pm = 0, 1/2$. However, as explained in [8], the solutions in these special cases are actually not acceptable, as they correspond to solutions of an equation with a δ-function source. This is a result of having removed the origin when we chose spherical coordinates. So, in all cases we set $B_2^\pm = 0 = C_2^\pm$. We also note here that the constants B_1^+, C_1^+ (B_1^-, C_1^-) are not independent

because the functions $\hat{F}^+, \hat{G}^+$ ($\hat{F}^-, \hat{G}^-$) are coupled through the first order differential equation (A21) [42]. In fact, one can obtain the consistency conditions

$$\frac{C_1^+}{B_1^+} = -\frac{2(2l+d-1)}{2l+d-2-2m-2\tilde{\omega}}\,, \qquad \frac{B_1^-}{C_1^-} = \frac{2(2l+d-1)}{2l+d-2+2m-2\tilde{\omega}}\,. \tag{B5}$$

Now consider the behavior near infinity, $x \to 0$. For this, it is best to write the hypergeometric functions as functions of $\sin^2 x$. For example, when $\nu_\pm \neq 0, 1, 2, \ldots$, we have

$$\begin{aligned}\hat{F}^\pm = \; & B_1^\pm \;\; (\cos x)^{\sigma_\pm + 1/2} (\sin x)^{-\nu_- + 1/2} \Bigg[\frac{\Gamma(1+\sigma_\pm)\Gamma(\nu_-)}{\Gamma(\zeta^{\tilde{\omega}}_{\nu_-,\sigma_\pm})\Gamma(\zeta^{-\tilde{\omega}}_{\nu_-,\sigma_\pm})} {}_2F_1(\zeta^{\tilde{\omega}}_{-\nu_-,\sigma_\pm}, \zeta^{-\tilde{\omega}}_{-\nu_-,\sigma_\pm}, 1-\nu_-, \sin^2 x) \\ & + \;\; \frac{\Gamma(1+\sigma_\pm)\Gamma(-\nu_-)}{\Gamma(\zeta^{\tilde{\omega}}_{-\nu_-,\sigma_\pm})\Gamma(\zeta^{-\tilde{\omega}}_{-\nu_-,\sigma_\pm})} (\sin x)^{2\nu_-} {}_2F_1(\zeta^{\tilde{\omega}}_{\nu_-,\sigma_\pm}, \zeta^{-\tilde{\omega}}_{\nu_-,\sigma_\pm}, 1+\nu_-, \sin^2 x) \Bigg]\,,\end{aligned} \tag{B6}$$

with the corresponding expression for $\hat{G}^\pm$ given by exchanging $\nu_- \to \nu_+, \sigma_\pm \to \sigma_\mp$. The transformations of the hypergeometric functions for the remaining cases of integer $\nu_\pm$ are given in [8]. Near the boundary, the leading terms in all cases are

$$\hat{F}^\pm \;\sim\; B_1^\pm \frac{\Gamma(1+\sigma_\pm)\Gamma(\nu_-)}{\Gamma(\zeta^{\tilde{\omega}}_{\nu_-,\sigma_\pm})\Gamma(\zeta^{-\tilde{\omega}}_{\nu_-,\sigma_\pm})} (\sin x)^{-\nu_- + 1/2} + \ldots \tag{B7}$$

$$\hat{G}^\pm \;\sim\; C_1^\pm \frac{\Gamma(1+\sigma_\mp)\Gamma(\nu_+)}{\Gamma(\zeta^{\tilde{\omega}}_{\nu_+,\sigma_\mp})\Gamma(\zeta^{-\tilde{\omega}}_{\nu_+,\sigma_\mp})} (\sin x)^{-\nu_+ + 1/2} + \ldots \tag{B8}$$

We examine the mass ranges with distinct behavior in turn.

$\mathbf{m \geq 3/2}$: This corresponds to $\nu_- \geq 1, \nu_+ \geq 2$. Then $\hat{F}^+$ is not square integrable unless $\Gamma(\zeta^{\tilde{\omega}}_{\nu_-,\sigma_+})$ or $\Gamma(\zeta^{-\tilde{\omega}}_{\nu_-,\sigma_+})$ diverges, i.e.

$$\tilde{\omega} = \mp(2n + 1 + \nu_- + \sigma_+)\,, \quad n = 0, 1, 2, \ldots\,. \tag{B9}$$

Since $\hat{F}^+, \hat{G}^+$ were coupled in the original Dirac equation, this also fixes $\tilde{\omega}$ in the $\hat{G}^+$ solution. Then $\Gamma(\zeta^{\pm\tilde{\omega}}_{\nu_+,\sigma_-}) = \Gamma(-n)$ diverges, and so this ensures that $\hat{G}^+$ is also square integrable. Similarly, $\hat{F}^-$ is not square integrable unless $\Gamma(\zeta^{\tilde{\omega}'}_{\nu_-,\sigma_-})$ or $\Gamma(\zeta^{-\tilde{\omega}'}_{\nu_-,\sigma_-})$ diverges, i.e.

$$\tilde{\omega}' = \mp(2n' + 1 + \nu_- + \sigma_-)\,, \quad n' = 1, 2, \ldots\,. \tag{B10}$$

(Here we have been careful to note that one could choose independent frequencies ω and ω' for the ψ^+ and ψ^- solutions.) Since $\hat{F}^-, \hat{G}^-$ were coupled in the original Dirac equation, this also fixes $\tilde{\omega}'$ in the $\hat{G}^-$ solution. Then $\Gamma(\zeta^{\pm\tilde{\omega}'}_{\nu_+,\sigma_+}) = \Gamma(-n'+1)$ diverges, and so this ensures that $\hat{G}^-$ is also square integrable.

$\mathbf{1/2 \leq m < 3/2}$: This corresponds to $0 \leq \nu_- < 1, 1 \leq \nu_+ < 2$. Then, $\hat{F}^+$ is square integrable for all $\tilde{\omega}$, but we still must fix

$$\tilde{\omega} = \mp(2n + 1 + \nu_+ + \sigma_-)\,, \quad n = 0, 1, 2, \ldots \tag{B11}$$

to ensure that $\hat{G}^+$ is square integrable. Similarly, $\hat{F}^-$ is square integrable for all $\tilde{\omega}'$, but we still must fix

$$\tilde{\omega}' = \mp(2n' + 1 + \nu_+ + \sigma_+)\,, \quad n' = 0, 1, 2, \ldots \tag{B12}$$

to ensure that $\hat{G}^-$ is square integrable.

$\mathbf{0 \leq m < 1/2}$: This corresponds to $0 < \nu_- \leq 1/2, 1/2 \leq \nu_+ < 1$. In this case, $\hat{F}^\pm, \hat{G}^\pm$ are square integrable for all $\tilde{\omega}$.

We have thus found that for $m \geq 1/2$, requiring the inner product to be finite imposes a unique boundary condition. In terms of the original spinor fields ψ, we note (using (A11)) that near infinity

$$\psi^\pm \sim \begin{pmatrix} i\mathscr{Y}^\pm \\ \mathscr{Y}^\mp \end{pmatrix} e^{-i\omega t} (\sin x)^{\frac{d-1}{2}} \hat{F}^\pm + \begin{pmatrix} -i\mathscr{Y}^\pm \\ \mathscr{Y}^\mp \end{pmatrix} e^{-i\omega t} (\sin x)^{\frac{d-1}{2}} \hat{G}^\pm\,, \quad d \text{ even} \tag{B13}$$

and similarly for d odd. Expanding $\hat{F}^{\pm}, \hat{G}^{\pm}$ for $x \to 0$ and using the frequency quantization conditions above, we find that asymptotically

$$\psi \sim \beta(\sin x)^{\frac{d-1}{2}+m} + O\left((\sin x)^{\frac{d+1}{2}+m}\right) , \tag{B14}$$

where the coefficient β is a spinor depending on time and angles on the S^{d-2}, but not on x.

For $0 \leq m < 1/2$, there will be a choice of boundary conditions at infinity. Note that, unlike the scalar case, this mass range does not depend on d. Using (B13) and expanding $\hat{F}^{\pm}, \hat{G}^{\pm}$ for $x \to 0$, we find that near infinity

$$\psi \sim \alpha(\sin x)^{\frac{d-1}{2}-m} + \beta(\sin x)^{\frac{d-1}{2}+m} + O\left((\sin x)^{\frac{d+1}{2}-m}\right) , \tag{B15}$$

where the coefficients α, β are spinors depending only on time and angles on the S^{d-2}. Using the properties of the spinor spherical harmonics under the action of the radial gamma matrix $x^k\gamma^{\hat{k}}$ (see appendix A), one can verify that $P_-\alpha = 0$, $P_+\beta = 0$ where we have defined the radial gamma matrix projectors $P_{\pm} = \frac{1}{2}\left(1 \pm \frac{1}{r}x^k\gamma^{\hat{k}}\right)$.

One may also work out the sub-leading terms which will be needed for the study of supersymmetry in the main text. The important step is to rewrite the Dirac equation in terms of the unphysical metric $\tilde{g}_{ab}$:

$$\gamma^a\nabla_a\psi - m\psi = \Omega\tilde{\gamma}^a\tilde{\nabla}_a\psi - \frac{d-1}{2}\,\tilde{n}_a\tilde{\gamma}^a\psi - m\psi = 0\,. \tag{B16}$$

If we now insert the expansion (2.7) into the above equation and collect terms we find

$$\begin{aligned} 0 = \; & - \; m\,(1+\tilde{n}_a\tilde{\gamma}^a)\,\alpha\Omega^{\frac{d-1}{2}-m} - m\,(1-\tilde{n}_a\tilde{\gamma}^a)\,\beta\Omega^{\frac{d-1}{2}+m} \\ & + \; \left[\tilde{\gamma}^a\tilde{\nabla}_a\alpha - (m+(m-1)\tilde{n}_a\tilde{\gamma}^a)\,\alpha'\right]\Omega^{\frac{d+1}{2}-m} \\ & + \; \left[\tilde{\gamma}^a\tilde{\nabla}_a\beta - (m-(m+1)\tilde{n}_a\tilde{\gamma}^a)\,\beta'\right]\Omega^{\frac{d+1}{2}+m} + O(\Omega^{\frac{d+3}{2}-|m|})\,. \end{aligned} \tag{B17}$$

Setting each term to zero leads to the relations (2.8),(2.9) stated in section II.

We conclude this appendix with a discussion of the relation to [8]. As noted in section II, an alternative method of analyzing allowed boundary conditions is to consider self-adjoint extensions of an appropriate spatial wave operator. In [8], this analysis is performed for massive scalars and massless vectors and tensors and it is shown that all such cases reduce to studying the operator

$$A = -\frac{d^2}{dx^2} + \frac{\nu^2 - 1/4}{\sin^2 x} + \frac{\sigma^2 - 1/4}{\cos^2 x} \tag{B18}$$

on the Hilbert space $L^2([0,\pi/2], dx)$. The results are determined by ν^2. If $\nu^2 < 0$, then A is unbounded below and so does not admit a positive extension; this is the case for scalars with $m^2 < m^2_{BF}$. If $\nu^2 \geq 0$, A is a positive operator (and therefore there exists at least one positive self-adjoint extension). For $\nu^2 \geq 1$ there is a unique self-adjoint extension that is automatically positive and so a unique linear boundary condition at infinity. However, for $0 \leq \nu^2 < 1$ (e.g. $m^2_{BF} \leq m^2_{\phi} < m^2_{BF} + 1$ for scalars) there is a family of such extensions corresponding to a choice of boundary conditions. Wald and Ishibashi proceed to determine all possible linear boundary conditions corresponding to positive self-adjoint extensions.

Since the same wave operator (B18) appears in (A23),(A24) and the inner product (B2) is the same as above, we can apply the analysis of [8] directly to any massive Dirac fermion: The wave operators are symmetric on the domain of smooth functions of compact support away from the origin, $C_0^{\infty}(0,\pi/2)$, and are positive for $\nu_{\pm}^2 \geq 0$, which implies that they admit at least one positive self-adjoint extension. In terms of the mass m, this condition is equivalent to

$$\left(m \pm \frac{1}{2}\right)^2 \geq 0 \Longrightarrow m^2 \geq 0\,. \tag{B19}$$

For $\nu_{\pm}^2 < 0$, the operators are unbounded below and therefore do not admit a positive self-adjoint extension. The inequality (B19) is the analogue of the Breitenlohner-Freedman bound for stability, though of course it is trivially satisfied for real m. The case $m^2 = 1/4$ is analogous to saturating the BF bound. For $m \geq 1/2$, the wave operators have a unique, positive self-adjoint extension and so there is a unique linear boundary condition at infinity. For $0 \leq m < 1/2$, there is a family of self-adjoint extensions, and a choice of boundary conditions at infinity. One could

also follow the von Neumann prescription as in [8] to classify positive self-adjoint extensions and the corresponding boundary conditions.

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